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DIFFERENTIAL EQUATIONS FROM THE ALGEBRAIC STANDPOINT

BY

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INTRODUCTION

We shall be concerned, in this monograph, with systems of differential equations, ordinary or partial, which are algebraic in the unknowns and their derivatives. The algebraic side of the theory of such systems seems to have remained, up to the present, in an undeveloped state.

It has been customary, in dealing with systems of differential equations, to assume canonical forms for the systems. Such forms are inadequate for the representation of general systems. It is true that methods have been proposed for the reduction of general systems to various canonical types. But the limitations which go with the use of the implicit function theorem, the lack of methods for coping with the phenomena of degeneration which are ever likely to occur in elimination processes and the absence of a technique for preventing the entrance of extraneous solutions, are merely symptoms of the futility inherent in such methods of reduction.

Now, in the theory of systems of algebraic equations, one witnesses a more enlivening spectacle. Kronecker's Festschrift of 1882 set upon a firm foundation the theory of algebraic elimination and the general theory of algebraic manifolds. The contributions of Mertens, Hilbert, König, Lasker, Macaulay, Henzelt, Emmy Noether, van der Waerden and others, have brought, to this division of algebra, a high degree of perfection. In the notions of irreducible manifold, and polynomial ideal, there has been material for far reaching qualitative and combinatorial investigations. On the formal side, one has universally valid methods of elimination and formulas for resultants.

To bring to the theory of systems of differential equations which are algebraic in the unknowns and their derivatives,

some of the completeness enjoyed by the theory of systems of algebraic equations, is the aim of the present monograph. The point of view which we take is that of our paper *Manifolds of functions defined by systems of algebraic differential equations*, published in volume 32 of the Transactions of the American Mathematical Society. In what follows, we shall outline our results.

Chapters I–VIII treat ordinary differential equations. We deal with any finite or infinite system of algebraic differential equations in the unknown functions $y_1, \cdots, y_n$ of the variable x. We write each equation in the form

$$F(x;\ y_1, \cdots, y_n) = 0,$$

where F is a polynomial in the y_i and any number of their derivatives. The coefficients in F will be supposed to be functions of x, meromorphic in a given open region. An expression like F, above, will be called a *form*. All forms considered in this introduction will be understood to have coefficients which are contained in a given *field*. By a field, we mean a set of functions which is closed with respect to rational operations and differentiation.*

Let Σ be any finite or infinite system of forms in $y_1, \cdots, y_n$. By a *solution* of Σ, we mean a solution of the system of equations obtained by setting the forms of Σ equal to zero. The totality of solutions of Σ will be called the *manifold* of Σ. If Σ_1 and Σ_2 are systems such that every solution of Σ_1 is a solution of Σ_2, we shall say that Σ_2 *holds* Σ_1.

A system Σ will be called *reducible* or *irreducible* according as there do or do not exist two forms, G and H, such that neither G nor H holds Σ, while GH holds Σ. The manifold of Σ, and also the system of equations obtained by equating the forms of Σ to zero, will be called reducible or irreducible according as Σ is reducible or irreducible.

We can now state the principal result of Chapter I. *Every manifold is composed of a finite number of irreducible manifolds.*

* A formal definition is given in § 1.

That is, given any system Σ, there exist a finite number of irreducible systems, $\Sigma_1, \cdots, \Sigma_s$, such that Σ holds every Σ_i, while every solution of Σ is a solution of some Σ_i. The decomposition into irreducible manifolds is essentially unique.

Let us consider an example. The equation

$$(1) \qquad \left(\frac{dy}{dx}\right)^2 - 4y = 0,$$

whose solutions are $y = (x-a)^2$, (a constant), and $y = 0$, is a reducible system in the field of all constants. For

$$(2) \qquad \frac{dy}{dx}\left(\frac{d^2y}{dx^2} - 2\right)$$

holds the first member of (1), while neither factor in (2) does. The system (1) is equivalent to the two irreducible systems

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0, \qquad \frac{dy}{dx} = 0$$

and

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0, \quad \frac{d^2y}{dx^2} - 2 = 0.$$

The decomposition theorem follows from a lemma which bears a certain analogy to Hilbert's theorem on the existence of a finite basis for an infinite system of polynomials. We prove that *if Σ is an infinite system of forms in $y_1, \cdots, y_n$, then Σ contains a finite subsystem whose manifold is identical with that of Σ.**

Chapters II and VI study irreducible manifolds. We start, in Chapter II, with a precise formulation of the notion of *general solution* of a differential equation. We do not think that such a formulation has been attempted before. Let A be a form in $y_1, \cdots, y_n$, effectively involving y_n, and irreducible, in the given field, as a polynomial in the y_i and

* See § 124 for a comparison, with a theorem of Tresse, of the extension of this lemma to partial differential equations.

their derivatives. Let the order of the highest derivative of y_n in A be r and let y_{nr} represent that derivative. Let Σ be the totality of forms which vanish for all solutions of A with $\partial A / \partial y_{nr} \neq 0$. We prove that Σ is irreducible. The manifold of Σ is one of the irreducible manifolds in the decomposition of the manifold of A. We call this manifold the *general solution of* A (or of $A = 0$).

The remainder of Chapter II deals with the association, with every irreducible system Σ, of a differential equation which we call a *resolvent* of Σ. The first member of the resolvent is an irreducible polynomial, so that the resolvent has a general solution. Roughly speaking, the determination of the general solution of the resolvent is equivalent to the determination of the manifold of Σ. The theory of resolvents furnishes a theoretical method for the construction of all irreducible systems. One will see that the resolvent can be used advantageously in formal problems.

In Chapter VI, we study what might be called the texture of an irreducible manifold. For the case of the general solution of an algebraically irreducible form, our work amounts to characterizing those singular solutions (solutions with $\partial A / \partial y_{nr} = 0$) which belong to the general solution.

Chapters V and VII contain, among other results, finite algorithms, involving differentiations and rational operations, for decomposing a finite system into irreducible systems and for constructing resolvents. In Chapter V, we do not obtain the actual irreducible systems, but rather certain *basic sets* of forms (Ch. II) which characterize the irreducible systems. However, this permits the construction of resolvents. In Chapter VII, a process is obtained which, if carried sufficiently far, will actually produce the irreducible systems. Unfortunately, there is nothing in this process which informs one, at any point, as to whether or not the process has had its desired effect.

The results of Chapter V furnish a complete elimination theory for systems of algebraic differential equations.

In Chapter VII, we derive an analogue, for differential forms, of the famous *Nullstellensatz* of Hilbert and Netto. In

Chapter VIII, we present an analogue of Lüroth's theorem on the parameterization of unicursal curves. In Chapter III, there will be found a theory of resultants of pairs of differential forms. A number of other special results are distributed through the monograph.

In Chapter X, some of the main results stated above are extended to systems of algebraic partial differential equations. In particular, an elimination theory is obtained for such systems.

Chapter IV treats systems of algebraic equations. The chief purpose is to obtain special theorems, and finite algorithms, for application to differential equation theory. The main results of Chapter IV are known ones, but the treatment appears new, and some special theorems, of importance for us, do not seem to exist in the literature.

It has been our aim to give this monograph an elementary character, and to assume only such facts of algebra and analysis as are contained in standard treatises. With this principle in mind, we have devoted Chapter IX to an exposition of Riquier's remarkable existence theorem for orthonomic systems of partial differential equations.

Thus Chapter IX is purely expository, and Chapter IV is largely so. The remaining chapters present results contained in our above mentioned paper, and results communicated by us to the American Mathematical Society since the publication of that paper.

Koenigsberger's irreducible differential equations,* and Drach's irreducible systems of partial differential equations,† are irreducible in the sense described above. In Drach's definition, which includes that of Koenigsberger, a system is called irreducible if every equation which admits one solution of the system admits all solutions of the system. Thus, systems which are irreducible in our sense may easily be reducible in the theories of Koenigsberger and Drach. The definitions of Koenigsberger and Drach, which do not

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* *Lehrbuch der Differenzialgleichungen*, Leipzig, 1889.

† Annales de l'Ecole Normale, vol. 34, (1898).

lead to decompositions into irreducible systems, are the starting points of group-theoretic investigations, which parallel the Galois theory. Our course, as we have seen, is in a different direction.

Many questions still remain for investigation. In particular, a theory of ideals of differential forms and a theory of birational transformations, await development.* Chapters VII and VIII may perhaps be regarded as rudimentary beginnings of such theories.

It goes without saying that we have been guided, in our work, by the existing theory of algebraic manifolds. We have found particularly valuable, the excellent treatment of systems of algebraic equations given in Professor van der Waerden's paper *Zur Nullstellentheorie der Polynomideale.*† But it is not surprising, on the other hand, that the investigation of essentially new phenomena should have called for the development of new methods.

I am very grateful to the Colloquium Committee of the American Mathematical Society, who have invited me to lecture on the subject of this monograph at the University of California in September, 1932. To my friend and colleague Dr. Eli Gourin, who assisted me in reading the proofs, I extend my deep thanks.

* In connection with transformations of general (non-algebraic) differential equations, see Hilbert, Mathematische Annalen, vol. 73 (1913), p. 95.

† Mathematische Annalen, vol. 96, (1927), p. 183.

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CONTENTS

CHAPTER I

DECOMPOSITION OF A SYSTEM OF ORDINARY ALGEBRAIC DIFFERENTIAL EQUATIONS INTO IRREDUCIBLE SYSTEMS

FIELDS

1. We consider functions meromorphic in a given *open region* $\mathfrak{A}$ in the plane of the complex variable x.* We recall that an open region is a set of points such that

(a) every point of the set is the center of a circle of positive radius, all of whose points belong to the set;

(b) any two points of the set can be joined by a continuous curve whose points all lie in the set.

A set $\mathfrak{F}$, of functions described as above, will be called a *field* if

(a) *$\mathfrak{F}$ contains at least one function which is not identically zero;*

(b) *given any two functions f and g (distinct or equal), belonging to $\mathfrak{F}$, then $f \pm g$ and fg belong to $\mathfrak{F}$;*

(c) *given any two functions, f and g, belonging to $\mathfrak{F}$, with g not identically zero, then f/g belongs to $\mathfrak{F}$;*

(d) *given any function, f, in $\mathfrak{F}$, the derivative of f belongs to $\mathfrak{F}$.*

Every field contains all rational constants. Examples of fields are: the totality of rational constants; the totality of rational functions of x; all rational combinations of x and e^x with constant coefficients; all elliptic functions with a given period parallelogram.†

* We are dealing here only with the finite plane.

† The notion of field of analytic functions has appeared previously, among other places, in Picard's group-theoretic investigations on linear

FORMS

2. In what follows, we work with an arbitrary field $\mathfrak{F}$, which is supposed to be assigned in advance and to stay fixed.

We are going to develop some notions in preparation for the study of differential equations in n unknown functions, $y_1, \cdots, y_n$.

By a *differential form* or, more briefly, by a *form*, we shall understand a polynomial in the y_i and any number of their derivatives, with coefficients meromorphic in $\mathfrak{A}$.

With respect to every form introduced into our work, we shall assume, unless the contrary is stated, that its coefficients belong to $\mathfrak{F}$.

Differentiation of functions y_i will be indicated by means of a second subscript. Thus

$$y_{ij} = \frac{d^j y_i}{dx^j}.$$

We write, frequently, $y_i = y_{i0}$.*

Throughout our work, capital italic letters will denote forms.

By the *jth derivative* of A, we mean the form obtained by differentiating A j times with respect to x, regarding $y_1, \cdots, y_n$ as functions of x.

By the *order of A with respect to y_i*, if A involves y_i or some of its derivatives effectively, we shall mean the greatest j

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differential equations and in Landau's work on the factorization of linear differential operators. See Picard, *Traité d'Analyse*, 2nd edition, vol. 3, p. 562. The foregoing writers make the additional assumption that $\mathfrak{F}$ contains all constants. Loewy, however, in his work on systems of linear differential equations, Mathematische Annalen, vol. 62 (1906), p. 89, does not make this additional assumption. No generality would be gained by allowing $\mathfrak{F}$ to consist of functions analytic except for isolated singularities. With this assumption, it is an easy consequence of Picard's theorem on essential singularities, and of the fact that $\mathfrak{F}$ contains all rational constants, that the functions in $\mathfrak{F}$ are meromorphic.

* In certain problems, we shall use unsubscripted letters to represent unknowns. If y is such an unknown, y_j will represent the jth derivative of y.

such that y_{ij} is present in a term of A with a coefficient distinct from zero. If A does not involve y_i, the order of A with respect to y_i will be taken as 0.

By the *class* of A, if A involves one or more y_i effectively, we shall mean the greatest p such that some y_{pj} is effectively present in A. If A is simply a function of x, A will be said to be of class 0.

Let A_1 and A_2 be two forms. If A_2 is of higher order than A_1 in some y_p, A_2 will be said to be of *higher rank than* A_1, and A_1 *of lower rank than* A_2, *in* y_p. If A_1 and A_2 are of the same order, say q, in y_p and if A_2 is of greater degree than A_1 in y_{pq},* then, again, A_2 will be said to be of higher rank than A_1 in y_p. Two forms for which no difference in rank is established by the foregoing criteria will be said to be of the same rank in y_p.

If A_2 is of higher class than A_1, A_2 will be said to be of *higher rank* than A_1.† If A_2 and A_1 are of same class $p > 0$, and if A_2 is of higher rank than A_1 in y_p, then, again, A_2 will be said to be of higher rank than A_1. Two forms for which no difference in rank is created by the preceding, will be said to be of the same rank.‡

If A_2 is higher than A_1, A_3 higher than A_2, then A_3 is higher than A_1.

In later chapters, we shall have occasion to use other symbols than $y_1, \cdots, y_n$ for the unknowns. If the unknowns are given in the order $u, v, \cdots, w$, then, in the definitions of class and of relative rank, the pth unknown from the left is to be treated like y_p above.

We shall need the following lemma:

LEMMA. *If*

$$A_1, A_2, \cdots, A_q, \cdots$$

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* Considered as a polynomial in y_{pq}. If a form is identically zero (hence of order zero in every y_p) it will be considered of degree 0 in every y_{p0}. This leads to no difficulties.

† We shall frequently say, simply, "A_2 is higher than A_1".

‡ Thus, all forms of class zero are of the same rank.

is an infinite sequence such that, for every q, A_{q+1} *is not higher than* A_q, *there exists a subscript* r, *such that, for* $q > r$, A_q *has the same rank as* A_r.

The classes of the A_q form a non-increasing set of non-negative integers. It is then clear that, for q large, the A_q have the same class, say p. If $p > 0$, the A_q with q large will be of the same order, say s, in y_p. Finally, the A_q will eventually have a common degree in y_{ps}.

An immediate consequence of this lemma is that *every finite or infinite aggregate of forms contains a form which is not higher than any other form of the aggregate.*

ASCENDING SETS

3. If A_1 is of class $p > 0$, A_2 will be said to be *reduced with respect to* A_1 if A_2 is of lower rank than A_1 in y_p.

The system

(1) $$A_1, A_2, \cdots, A_r$$

will be called an *ascending set* if either

(a) $r = 1$ *and* $A_1 \neq 0$

or

(b) $r > 1$, A_1 *is of class greater than* 0, *and, for* $j > i$, A_j *is of higher class than* A_i *and reduced with respect to* A_i.

Of course, $r \leqq n$.

The ascending set (1) will be said to be of *higher rank* than the ascending set

(2) $$B_1, B_2, \cdots, B_s$$

if either

(a) *There is a* j, *exceeding neither* r *nor* s, *such that* A_i *and* B_i *are of the same rank for* $i < j$ *and that* A_j *is higher than* B_j*

or

(b) $s > r$ *and* A_i *and* B_i *are of the same rank for* $i \leqq r$.

Two ascending sets for which no difference in rank is created by what precedes will be said to be of the same

* If $j = 1$, this is to mean that A_1 is higher than B_1.

rank. For such sets, $r = s$ and A_i and B_i are of the same rank for every i.

Let Φ_1, Φ_2, Φ_3 be ascending sets such that Φ_1 is higher than Φ_2, Φ_2 higher than Φ_3. We write $\Phi_1 > \Phi_2$, $\Phi_2 > \Phi_3$. We shall prove that $\Phi_1 > \Phi_3$.

Let Φ_1 and Φ_2 be represented by (1) and (2) respectively and let Φ_3 be

$$C_1, C_2, \cdots, C_t.$$

Suppose first that $\Phi_1 > \Phi_2$ for the reason (a) and that $\Phi_2 > \Phi_3$ for the reason (a). Let j be the smallest integer such that B_j is higher than C_j. Then either A_i is of the same rank as B_i for $i \leqq j$ or there is a $k \leqq j$ such that A_i is of the same rank as B_i for $i < k$ but that A_k is higher than B_k. In either case, $\Phi_1 > \Phi_3$ by (a).

Suppose now that $\Phi_1 > \Phi_2$ by (b), while $\Phi_2 > \Phi_3$ by (a). Let j be taken as above. If $j > r$, $\Phi_1 > \Phi_3$ by (b). If $j \leqq r$, $\Phi_1 > \Phi_3$ by (a).

Now let $\Phi_1 > \Phi_2$ by (a), while $\Phi_2 > \Phi_3$ by (b). Let j be the smallest integer for which A_j is higher than B_j. Then A_j is higher than C_j and A_i is of the same rank as C_i for $i < j$. Thus $\Phi_1 > \Phi_3$ through (a).

Finally, if $\Phi_1 > \Phi_2$ by (b) and $\Phi_2 > \Phi_3$ by (b), then $\Phi_1 > \Phi_3$ by (b).

We shall need the following fact:

Let

$$\Phi_1, \Phi_2, \cdots, \Phi_q, \cdots \tag{3}$$

be an infinite sequence of ascending sets such that Φ_{q+1} is not higher than Φ_q for any q. Then there exists a subscript r such that, for $q > r$, Φ_q has the same rank as Φ_r.

By the lemma of § 2, the first forms of the Φ_q (A_1 in (1)) are all of the same rank for q large. This accounts for the case in which Φ_q with q large has only one form. We may thus limit ourselves to the case in which Φ_q with q large has at least two forms. The second forms will eventually be of the same rank. Continuing, we find, since no Φ_q has more than n forms, that the Φ_q with q large all have the

same number of forms, corresponding forms being of the same rank. This proves the lemma.

An immediate consequence of this result is that *every finite or infinite aggregate of ascending sets contains an ascending set whose rank is not higher than that of any other ascending set in the aggregate.*

BASIC SETS

4. Let Σ be any finite or infinite system of forms, not all zero. There exist ascending sets in Σ; for instance, every non-zero form of Σ is an ascending set. Among all ascending sets in Σ, there are, by the final remark of § 3, certain ones which have a least rank. Any such ascending set will be called a *basic set* of Σ.

The following method for constructing a basic set of Σ can actually be carried out when Σ is finite. Of the non-zero forms in Σ, let A_1 be one of least rank. If A_1 is of class zero, it is a basic set for Σ. Let A_1 be of class greater than zero. If Σ contains no non-zero forms reduced with respect to A_1, then A_1 is a basic set. Suppose that such reduced forms exist; they are all of higher class than A_1. Let A_2 be one of them of least rank. If Σ has no non-zero forms reduced with respect to A_1 and A_2, then A_1, A_2 is a basic set. If such reduced forms exist, let A_3 be one of them of least rank. Continuing, we arrive at a set (1) which is a basic set for Σ.

If A_1, in (1), is of class greater than zero, a form F will be said to be *reduced with respect to the ascending set* (1) if F is reduced with respect to every A_i, $i = 1, \cdots, r$.

Let Σ be a system for which (1), with A_1 not of class zero, is a basic set. Then *no non-zero form of Σ can be reduced with respect to* (1). Suppose that such a form, F, exists. Then F must be higher than A_1, else F would be an ascending set lower than (1). Similarly, F must be higher than A_2, else A_1, F would be an ascending set lower than (1). Finally, F is higher than A_r. Then $A_1, \cdots, A_r, F$ is an ascending set lower than (1). This proves our statement.

Let Σ be as above. We see that *if a non-zero form, reduced with respect to* (1), *is adjoined to* Σ, *the basic sets of the resulting system are lower than* (1).

Throughout our work, large Greek letters not used as symbols of summation will denote systems of forms.

REDUCTION

5. In this section, we deal with an ascending set (1) with A_1 of class greater than 0.

If a form G is of class $p>0$, and of order m in y_p, we shall call the form $\partial G/\partial y_{pm}$ the *separant* of G. The coefficient of the highest power of y_{pm} in G will be called the *initial* of G.*

The separant and initial of G are both lower than G.

In (1), let S_i and I_i be respectively the separant and initial of A_i, $i=1,\cdots,r$.

We shall prove the following result.

Let G be any form. There exist non-negative integers s_i, t_i, $i=1,\cdots,r$, such that when a suitable linear combination of the A_i and of a certain number of their derivatives, with forms for coefficients, is subtracted from

$$S_1^{s_1}\cdots S_r^{s_r}I_1^{t_1}\cdots I_r^{t_r}G,$$

the remainder, R, is reduced with respect to (1).

We may limit ourselves to the case in which G is not reduced with respect to (1).

Let A_i be of class p_i, and of order m_i in y_{p_i}, $i=1,\cdots,r$.

Let j be the greatest value of i such that G is not reduced with respect to A_i. Let G be of order h in y_{p_j}.

We suppose first that $h>m_j$. If $k_1=h-m_j$, then $A_j^{(k_1)}$, the k_1th derivative of A_j, will be of order h in y_{p_j}. It will be linear in y_{p_jh}, with S_j for coefficient of y_{p_jh}. Using the algorithm of division, we find a non-negative integer v_1 such that

* Later we shall have occasion to use other symbols than $y_1,\cdots,y_n$ for unknowns. If the unknowns in a problem are given listed in the order $u,v,\cdots,w$, then w will play the role of y_p, above, in the definitions of separant and initial for a form effectively involving w.

2

$$S_j^{v_1} G = C_1 A_j^{(k_1)} + D_1$$

where D_1 is of order less than h in y_{p_j}. In order to have a unique procedure, we take v_1 as small as possible.

Suppose, for the moment, that $p_j < n$. Let a be an integer with $p_j < a \leqq n$. We shall show that D_1 is not of higher rank than G in y_a. We may limit ourselves to the case in which $D_1 \neq 0$. Also since S_j is free of y_a, we need treat only the case in which y_a is actually present in G. Let G be of order g in y_a. Then the order of D_1 in y_a cannot exceed g. If D_1 were of greater degree than G in y_{ag}, C_1 would have to involve y_{ag} in the same degree as D_1 and $C_1 A_j^{(k_1)}$ would contain terms involving y_{ag} and $y_{p_j h}$ which could be balanced neither by D_1 nor by $S_j^{v_1} G$. This proves our statement.

If D_1 is of order greater than m_j in y_{p_j}, we find a relation

$$S_j^{v_2} D_1 = C_2 A_j^{(k_2)} + D_2$$

with D_2 of lower order than D_1 in y_{p_j} and not of higher rank than D_1 (or G) in any y_a with $a > p_j$. For uniqueness, we take v_2 as small as possible.

Continuing, we eventually reach a D_u, of order not greater than m_j in y_{p_j}, such that, if

$$s_j = v_1 + \cdots + v_u,$$

we have

$$S_j^{s_j} G = E_1 A_j^{(k_1)} + \cdots + E_u A_j^{(k_u)} + D_u. \tag{4}$$

Furthermore, if $a > p_j$, D_u is not of higher rank than G in y_a.

If D_u is of order less than m_j in y_{p_j}, D_u is reduced with respect to A_j (as well as any A_i with $i > j$). If D_u is of order m_j in y_{p_j}, we find, with the algorithm of division, a relation

$$I_j^{t_j} D_u = H A_j + K$$

with K reduced with respect to A_j, as well as $A_{j+1}, \cdots, A_r$. For uniqueness, we take t_j as small as possible.

We now treat K as G was treated. For some $l < j$, there are s_l, t_l such that $S_l^{s_l} I_l^{t_l} K$ exceeds, by a linear combination of A_l and its derivatives, a form L which is reduced with respect to $A_l, A_{l+1}, \cdots, A_r$. Then

$$S_l^{s_l} S_j^{s_j} I_l^{t_l} I_j^{t_j} G$$

exceeds L by a linear combination of A_l, A_j and their derivatives.

Continuing, we reach a form R as described in the statement of the lemma.

Our procedure determines a *unique* R. We call this R the *remainder of* G *with respect to the ascending set* (1).

Solutions and manifolds

6. Let Σ represent any finite or infinite system. The forms in Σ need not all be distinct from one another.*

When the forms of Σ are equated to zero, we obtain a system of differential equations, which we shall represent symbolically by $\Sigma = 0$.

In studying the totality of solutions of $\Sigma = 0$, it will be of fundamental importance to have a sharp definition of *solution*. Let $y_1, \cdots, y_n$ be functions, analytic throughout an open region $\mathfrak{B}$, whose points are in $\mathfrak{A}$, which render each form of Σ zero when substituted into the form. The entity composed of $\mathfrak{B}$ and of $y_1, \cdots, y_n$ will be called a *solution* of $\Sigma = 0$. Thus two systems $y_1, \cdots, y_n$ which are identical from the point of view of analytic continuation, will give different solutions if they are not associated with the same open region. For instance, if we take an open region $\mathfrak{B}_1$, interior to $\mathfrak{B}$, and use, throughout $\mathfrak{B}_1$, $y_1, \cdots, y_n$ as defined for $\mathfrak{B}$, we get a second solution of $\Sigma = 0$.†

* What we are really considering then, is a system of marks, each mark being associated with a form. Two distinct marks may be associated with identical forms.

† In Chapter VII, we shall, at one point, adopt a different definition, calling any set of n formal power series, convergent or divergent, a solution, if they yield zero when substituted formally into the forms of Σ. Many of our results hold for this definition.

By a *solution of* Σ, we shall mean a solution of $\Sigma = 0$. The totality of solutions of Σ will be called the *manifold* of Σ (or of $\Sigma = 0$).

If Σ_1 and Σ_2 are systems such that every solution of Σ_1 is a solution of Σ_2, we shall say that Σ_2 *holds* Σ_1.*

COMPLETENESS OF INFINITE SYSTEMS

7. In §§ 7—10, we prove the following lemma:

LEMMA. *Every infinite system of forms in* $y_1, \cdots, y_n$ *has a finite subsystem whose manifold is identical with that of the infinite system.*†

An infinite system of forms whose manifold is identical with that of one of its finite subsystems will be called *complete*.‡ Infinite systems which are not complete will be called *incomplete.* In what follows, we assume the existence of incomplete systems, and force a contradiction.

8. The system obtained by adjoining forms $G_1, \cdots, G_m$ to a system Σ will be denoted by $\Sigma + G_1 + \cdots + G_m$.

We prove the following lemma:

LEMMA: *Let* Σ *be an incomplete system. Let* $F_1, \cdots, F_s$ *be such that, by multiplying each form in* Σ *by some product of non-negative powers of* $F_1, \cdots, F_s$, *a system* Λ *is obtained which is complete.*§ *Then* $\Sigma + F_1 F_2 \cdots F_s$ *is incomplete.*

Let $\Sigma + F_1 \cdots F_s$ be complete, and let it hold and be held by its finite subset

$$(5) \qquad F_1 \cdots F_s, \qquad H_1, \cdots, H_t.$$

The presence of $F_1 \cdots F_s$ in (5) is legitimate, for if $\Sigma + F_1 \cdots F_s$ has the same manifold as a system Γ, it has the same manifold as $\Gamma + F_1 \cdots F_s$.

Let

$$(6) \qquad K_1, \cdots, K_v$$

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* If Σ_1 has no solutions, every system will be said to hold Σ_1.

† See § 124 for a comparison of this lemma with a result of Tresse.

‡ If some finite subsystem has no solutions, the system will be considered complete.

§ The product of powers of $F_1, \cdots, F_s$ may, of course, be different for different forms of Σ.

be forms in Σ such that the forms of Λ which they yield, after the above described multiplications, form a system Φ which is held by Λ. If some K_i are not among the H_i in (5) we may, as was seen above, adjoin them to the H_i. Similarly, any H_i not present in (6) may be adjoined to (6). We shall thus assume that (6) is identical with

$$H_1, \cdots, H_t. \tag{7}$$

Let L, in Σ, not hold (7). Now some

$$F_1^{g_1} \cdots F_s^{g_s} L$$

holds Φ, and Φ holds (7). Then $F_1 \cdots F_s L$ holds (7). Consequently certain solutions of $F_1 \cdots F_s$ which are solutions of (7) are not solutions of L. Thus L does not hold (5). This proves the lemma.

9. We prove the following lemma:

LEMMA. *Let Σ and $\Sigma + F_1 \cdots F_s$ both be incomplete. Then at least one of the systems $\Sigma + F_1, \cdots, \Sigma + F_s$ is incomplete.*

We may evidently limit ourselves to the case of $s = 2$. Let $\Sigma + F_1$ and $\Sigma + F_2$ both be complete. Let Φ_i, $i = 1, 2$, be a finite subset of Σ such that $\Sigma + F_i$ holds $\Phi_i + F_i$.

Then $\Sigma + F_1$ holds $(\Phi_1 + \Phi_2) + F_1$ and $\Sigma + F_2$ holds $(\Phi_1 + \Phi_2) + F_2$.* Now every solution of $(\Phi_1 + \Phi_2) + F_1 F_2$ is a solution of $(\Phi_1 + \Phi_2) + F_1$ or a solution of $(\Phi_1 + \Phi_2) + F_2$. As $\Sigma + F_1 F_2$ holds $\Sigma + F_1$ and $\Sigma + F_2$, then $\Sigma + F_1 F_2$ holds $(\Phi_1 + \Phi_2) + F_1 F_2$. This proves the lemma.

10. Let us consider the totality of incomplete systems of forms in $y_1, \cdots, y_n$. According to the final remark of § 3, there is one of them, Σ, whose basic sets (§ 4) are not higher than those of any other incomplete system. Let (1) be a basic set of Σ. Then A_1 involves unknowns, else A_1 would have no solutions, and Σ would be complete.

For every form of Σ not in (1), let a remainder with respect to (1) be found as in § 5. Let Λ be the system composed of the forms of (1) and of the products of the forms

* $(\Phi_1 + \Phi_2)$ consists of the forms present either in Φ_1 or in Φ_2.

of Σ not in (1) by the products $S_1^{s_1} \cdots I_r^{t_r}$ used in their reduction. Let Ω be the system composed of (1) and of the remainders of the forms of Σ not in (1).

Now Ω must be complete. If not, it would certainly have non-zero forms not in (1). Since such forms would be reduced with respect to (1), then (1) could not be a basic set of Ω (§ 4). This means that Ω would have ascending sets, hence basic sets, lower than (1) and Σ would not be an incomplete system with lowest basic sets.

If H is a form of Δ not in (1), and R the corresponding form in Ω, then H and R have the same solutions in common with (1). This means that Δ and Ω have the same manifold and also that Δ is complete.

The lemmas of §§ 8, 9 show us now that either some $\Sigma + S_i$ is incomplete or some $\Sigma + I_i$ is incomplete. But, for every i, S_i and I_i are distinct from zero, and reduced with respect to (1). Then, by § 4, the basic sets of $\Sigma + S_i$ and of $\Sigma + I_i$ are of lower rank than (1). This proves the fundamental lemma stated in § 7.

Non-existence of a Hilbert theorem

11. One might conjecture, on the basis of Hilbert's theorem relative to the existence of finite bases for infinite systems of polynomials,* that, in every infinite system Σ, there is a finite system such that every form of Σ is a linear combination of the forms of the finite system, and their derivatives, with forms for coefficients. We shall show that this is not so.

We consider forms in a single unknown y. (See first footnote in § 2.)

Consider the system

$$y_1 y_2, \quad y_2 y_3, \quad \cdots, \quad y_n y_{n+1}, \quad \cdots.$$

We shall show that no form of this system with $n > 1$ is linearly expressible in terms of the forms which precede it, and their derivatives.

* van der Waerden, *Moderne Algebra*, vol. 2, p. 23.

We notice that all of the forms, and all of their derivatives, are homogeneous polynomials of the second degree in the y_i. Also, if the weight of $y_i y_j$ is defined as $i+j$, the pth derivative of $y_i y_j$ will be isobaric, with its terms of weight $i+j+p$.

Now if

$$y_n y_{n+1} = A_1 y_1 y_2 + \cdots + A_{n-1} y_{n-1} y_n + B_1 \frac{d}{dx}(y_1 y_2) + \cdots,$$

the terms in the A_i, B_i, etc., which are not independent of the y_i may be cast out, for they produce terms of degree greater than 2. Again, considering the weights of the various forms, we find that

$$(8) \qquad y_n y_{n+1} = C_1 \frac{d^{2n-2}}{dx^{2n-2}}(y_1 y_2) + \cdots + C_{n-1} \frac{d^2}{dx^2}(y_{n-1} y_n),$$

with C_i which are independent of the y_i. Now the $(2n-2)d$ derivative of $y_1 y_2$ contains a term $y_1 y_{2n}$, and none of the other derivatives in (8) yields such a term. We conclude that $C_1 = 0$. Continuing, we find every C_i to be zero. This proves our statement.

Irreducible systems

12. A system Σ will be said to be *reducible* if there exist two forms, G and H such that neither G nor H holds Σ but that GH holds Σ. Systems which are not reducible will be called *irreducible*. The system of equations $\Sigma = 0$, and also the manifold of Σ, will be said to be *reducible* or *irreducible* according as Σ is reducible or irreducible.

Example 1. Let Σ, in the unknown y, consist of $y_1^2 - 4y$ and $y_2 - 2$. (See Introduction, p. v.) Let GH hold Σ. Let G_1 and H_1 be the remainders for G and H respectively with respect to $y_2 - 2$. Then G_1 and H_1 will be at most of order 1 and $G_1 H_1$ holds Σ. Then as every $y = (x-a)^2$ with a constant is a solution of Σ, $G_1 H_1$, if not zero, must be of order 1. Let K be the remainder of $G_1 H_1$ with respect to $y_1^2 - 4y$. One can prove now without difficulty that K vanishes identically. Then $G_1 H_1$ is algebraically divisible

by y_1^2-4y. As y_1^2-4y is algebraically irreducible,* one of G_1, H_1 must be divisible by y_1^2-4y. This means, since the initial and separant of y_2-2 are both unity, that one of G, H is a linear combination of the two forms of Σ and their derivatives. Then Σ is irreducible in every field.

Example 2. We use two unknowns, u and y. Let $\Sigma=0$ be $uy-u_1^2=0$. Differentiating, we find

$$u_1 y + u y_1 - 2u_1 u_2 = 0.$$

Multiplying the last equation through by y and using $\Sigma=0$, we have

$$u_1 y^2 + u_1^2 y_1 - 2u_1 u_2 y = 0.$$

Certainly u_1 does not hold Σ. Neither does

$$y^2 + u_1 y_1 - 2u_2 y,$$

since it vanishes only for $y=0$, if $u=0$. Thus Σ is reducible in the field of rational constants. We call attention to the fact that $uy-u_1^2$ is algebraically irreducible, and of order 0 in y.

THE FUNDAMENTAL THEOREM

13. A system Σ will be said to be *equivalent to the set of systems* $\Sigma_1, \cdots, \Sigma_s$ if Σ holds every Σ_i and every solution of Σ is a solution of some Σ_i. Thus, two systems with the same manifold are equivalent to each other.

We prove the following fundamental theorem.

THEOREM. *Every system of forms is equivalent to a finite of irreducible systems.*

Let the theorem be false for some Σ. Then Σ is reducible. Let G_1 and G_2 be such that $G_1 G_2$, but neither G_1 nor G_2, holds Σ. Then Σ is equivalent to the set

$$\Sigma + G_1, \qquad \Sigma + G_2. \tag{9}$$

Thus at least one of the systems (9) is reducible. A reducible system in (9) will be called a system of the *first class.*

* That is, irreducible as a polynomial in y_1, y.

There must be a system of the first class, which, when treated like Σ, yields one or two reducible systems obtained by adjoining two forms to Σ. The reducible systems obtained through two adjunctions, we call systems of the *second class.* Some of the systems of the second class, when treated like Σ, must yield reducible systems obtained from Σ by three adjunctions, that is, systems of the *third class.* We proceed in this manner, forming systems of all classes.

There must be a system of the first class whose forms are contained in systems of all classes higher than the first. Let $\Sigma+H_1$, where H_1 is either G_1 or G_2, be such a system of the first class. One of the systems of the second class which contains the forms of $\Sigma+H_1$ must have its forms contained in systems of all classes higher than the second. Let $\Sigma+H_1+H_2$ be such a system. Let an H_p be found, in this way, for every p. Then the system Ψ, composed of

$$\Sigma,\ H_1,\ H_2,\ \cdots,\ H_p,\ \cdots$$

is incomplete. For, if Ψ held

$$\Phi+H_{i_1}+\cdots+H_{i_q}$$

with Φ a finite subsystem of Σ and $i_1<\cdots<i_q$, then Ψ would hold

$$\Sigma+H_1+\cdots+H_{i_q}. \tag{10}$$

This cannot be, since H_{i_q+1} does not hold (10). This proves our theorem. One will notice that the proof involves making an infinite number of selections.*

Uniqueness of decomposition

14. Let a system Σ be equivalent to the set of irreducible systems

$$\Sigma_1,\ \cdots,\ \Sigma_s. \tag{11}$$

We may suppose, suppressing certain of the Σ_i if necessary, that no Σ_i holds a Σ_j with $j\neq i$. We shall then call each Σ_i

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* See § 70.

an *essential irreducible system held by* Σ, and we shall call (11) a *decomposition of* Σ *into essential irreducible systems.*

We shall prove that *the decomposition* (11) *of* Σ *into essential irreducible systems is essentially unique.* That is, *if* $\Omega_1, \cdots, \Omega_t$ *is a second decomposition of* Σ *into essential irreducible systems, then* $t = s$ *and every* Ω_i *is equivalent to some* Σ_j.

We shall show that there is some Ω_i which holds Σ_1. If there were not, then each Ω_i would have a form which would not hold Σ_1. Such forms being selected, their product would hold each Ω_i, consequently Σ, thus Σ_1. This is impossible if Σ_1 is irreducible and none of the forms holds Σ_1.

Then let Ω_1 hold Σ_1. Now Ω_1, similarly, must be held by some Σ_i, which must be Σ_1, since no Σ_i with $i \neq 1$ holds Σ_1. Thus Σ_1 und Ω_1 are equivalent. The uniqueness is proved.

EXAMPLES

15. We shall consider some examples involving one unknown, y, in which, in spite of the fact that the systems decomposed consist of a single form, the results are not uninstructive.

Example 1. Let $\Sigma = 0$ be $y_2^2 - y = 0$. By differentiation, we find, for any solution of Σ,

$$2y_2y_3 - y_1 = 0,$$

$$2y_2y_4 + 2y_3^2 - y_2 = 0, \tag{12}$$

$$2y_2y_5 + 6y_3y_4 - y_3 = 0. \tag{13}$$

Multiplying (13) by $2y_3$ and substituting into the result the expression for y_3^2 found from (12), we find that

$$y_2(4y_3y_5 - 12y_4^2 + 8y_4 - 1) = 0.$$

Thus Σ is equivalent to the set of two systems

$$\Sigma_1: \quad y_2^2 - y, \quad y_2;$$

$$\Sigma_2: \quad y_2^2 - y, \quad 4y_3y_5 - 12y_4^2 + 8y_4 - 1.$$

As the only solution of Σ_1 is $y = 0$, Σ_1 is irreducible in every field. We shall see in the next chapter that the

manifold of Σ_2, which is the "general solution" of $y_2^2 - y$, is irreducible in every field. We note that Σ_2 does not hold Σ_1.

Example 2. Let $\Sigma = 0$ be $y_1^2 y_2 - y = 0$. We find, with a single differentiation, that Σ is equivalent to the two systems:

$$\Sigma_1: \quad y_1^2 y_2 - y, \qquad y_1;$$
$$\Sigma_2: \quad y_1^2 y_2 - y, \qquad y_1 y_3 + 2y_2^2 - 1.$$

Σ_1 and Σ_2 are irreducible in any field (as above). We call attention to the fact that the form in Σ is linear in y_2.

Example 3. The form $y_1(y_1 - y)$ decomposes into the essential irreducible systems y_1 and $y_1 - y$. These two systems have the solution $y = 0$ in common.

The above examples might lead one to conjecture that any Σ can be decomposed into irreducible systems by means of differentiation and elimination. We shall see in Chapter VII that this is actually so.

Relative reducibility

16. Let A be any system of forms. A system Σ will be said to be *reducible relatively* to A if there exist forms G and H in A such that GH, but neither G nor H, holds Σ. Otherwise Σ will be said to be *irreducible relatively* to A.

For instance, if Σ is the form $(dy/dx)^2 - 4y$, Σ is reducible in the field of rational constants if A is the set of all forms in y of orders 0, 1, 2, but is irreducible in any field if A is the set of all forms of orders 0, 1. (See example 1, § 12.)

We see, as in § 13, that every system is equivalent to a finite number of systems irreducible relatively to A.

The decomposition into relatively irreducible systems need not be unique. For instance if A is the form 1, the system in the above example, which is relatively irreducible, is equivalent to the two relatively irreducible systems $y_1^2 - 4y$, y_1 and $y_1^2 - 4y$, $y_2 - 2$.

If Σ consists of forms in $\mathcal{A}$, Σ can be resolved into relatively irreducible systems whose forms belong to $\mathcal{A}$. If $\mathcal{A}$ is such that the product of two forms of $\mathcal{A}$ belongs to $\mathcal{A}$, such a decomposition is essentially unique in the sense of § 14.

Wherever the contrary is not stated, we shall deal with irreducibility as defined in § 12. That is $\mathcal{A}$ will consist of all forms with coefficients in $\mathfrak{F}$.

Adjunction of new unknowns

17. One might ask how the theory of a system Σ in the unknowns $y_1, \cdots, y_n$ is affected if new unknowns $v_1, \cdots, v_t$ are introduced, and Σ is regarded as a system of forms in the y_i, v_i. For instance, will the decomposition (11) of Σ into irreducible systems, when the y_i are the unknowns, continue to be such a decomposition when the unknowns are the y_i and v_i?

To show that the answer to this question is affirmative, we consider an irreducible system Σ of forms in the y_i and prove that it remains irreducible when the unknowns are the y_i, v_i. We represent Σ, considered as a system in the y_i, v_i, by Σ'.

Suppose that G and H are forms in the y_i, v_i such that neither holds Σ', but that GH holds Σ'. Let G and H be arranged as polynomials in the v_{ij}, with coefficients which are forms in the y_i.

We note that the solutions of Σ' are obtained by adjoining, to every solution $y_1, \cdots, y_n$ of Σ, arbitrarily assigned functions $v_1, \cdots, v_t$.

Evidently, then, the terms of G and H in which the coefficients hold Σ can be suppressed and the modified G and H will be such that neither holds Σ', while GH does. We assume thus that no coefficient in G or H holds Σ.

As Σ is irreducible, it will have a solution for which no coefficient in G or H vanishes. Then we can certainly replace the v_{ij}, in G and H, by rational constants, so as to get two forms, G_1 and H_1, in the y_i, neither of which holds Σ. On

the other hand, since we can construct analytic functions v_i for which the v_{ij} in GH have any assigned values, at any given point, and since GH holds Σ', it is necessary that $G_1 H_1$ hold Σ. This proves that Σ' is irreducible.

Fields of constants

18. In later work, it will at times be desirable to assume that $\mathfrak{F}$ contains at least one function which is not a constant. We establish now a result which will permit us to make this assumption with no real loss of generality.

Suppose that $\mathfrak{F}$ consists purely of constants. Let $\mathfrak{F}_1$ be the field obtained by adjoining x to $\mathfrak{F}$, that is, the totality of rational functions of x with coefficients in $\mathfrak{F}$. We shall prove that if a system Σ of forms in $\mathfrak{F}$ is irreducible in $\mathfrak{F}$, then Σ is irreducible in $\mathfrak{F}_1$.

We start by proving that if G, of the type

$$B_0 + B_1 x + \cdots + B_m x^m, \tag{14}$$

with the B_i forms in $\mathfrak{F}$, holds Σ, then each B_i holds Σ. Let

$$y_1(x), \cdots, y_n(x) \tag{15}$$

be any solution of Σ. Since the forms in Σ have constant coefficients,

$$y_1(x+c), \cdots, y_n(x+c)$$

where c is a small constant, will also be a solution of Σ.* This means that, for any solution (15),

$$B_0 + B_1 \xi + \cdots + B_m \xi^m$$

where ξ is any constant, vanishes identically in x. Then each B_i must vanish identically in x. This proves our statement.

* We shall not encumber our discussions with references to the areas in which the solutions are analytic.

Now, let G and H be forms in $\mathfrak{F}_1$ such that GH holds Σ. We have to prove that one of G, H holds Σ. We may evidently limit ourselves to the case in which G is given by (14) and H by

$$C_0 + C_1 x + \cdots + C_s x^s$$

with the C_i forms in $\mathfrak{F}$.

Suppose that neither G nor H holds Σ. In G and H, let every B_i and C_i which holds Σ be suppressed. For the modified G and H, GH will still hold Σ. Then

$$GH = \cdots + B_m C_s x^{m+s}.$$

Since neither B_m nor C_s holds Σ, $B_m C_s$ cannot hold Σ, so that GH cannot hold Σ. This proves that Σ is irreducible in $\mathfrak{F}_1$.

CHAPTER II

GENERAL SOLUTIONS AND RESOLVENTS

GENERAL SOLUTION OF A DIFFERENTIAL EQUATION

19. We consider a form A in $y_1, \cdots, y_n$ of class n, which is *algebraically irreducible* in $\mathfrak{F}$, that is, not the product of two forms, each of class greater than 0, and each with coefficients in $\mathfrak{F}$.

We are going to introduce the notion of the *general solution* of A.

We write $y_n = y$ and, if $n > 1$, we write $q = n - 1$, $y_i = u_i$, $i = 1, \cdots, q$.

Our definition of the general solution will appear, at first, to depend on the order in which the unknowns happen to be arranged,* at least, on the manner in which y is selected from among the unknowns effectively present in A. But it will turn out, finally, that the object which we define is actually independent of such order.

20. Let S and I be, respectively, the separant and initial of A.† A solution of A for which neither S nor I vanishes will be called a *regular* solution of A.

We shall make plain that regular solutions of A exist. Let A be of order s in y. Since SI is of lower degree than A in y_s, SI and A, considered as polynomials in the unknowns and their derivatives, are relatively prime. Then‡ there is a $B \neq 0$ which, if $s > 0$, is of order less than s in y and which, if $s = 0$, is free of y, such that

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* That is on the manner in which the subscripts $1, \cdots, n$ are attributed to the unknowns.

† See footnote in § 5.

‡ Bocher, *Algebra*, p. 213; Perron, *Algebra*, vol. 1, p. 204.

$$(1) \qquad B = C(SI) + DA.$$

We shall use the symbol ξ to designate values of x at which all coefficients of the forms in (1) are analytic, and the symbol $[\eta]$ to represent any set of numerical values which one may choose to attribute to the unknowns and their derivatives present in A, omitting y_s. Let ξ, $[\eta]$ be taken so that $IB \neq 0$. We can then find a number ζ such that $A = 0$ for $y_s = \zeta$, when the other symbols in A are replaced by their values $\xi[\eta]$. Then, by (1), SI cannot vanish for ξ, $[\eta]$, ζ.

In particular, since $S \neq 0$, we see by the implicit function theorem that there exists a function

$$(2) \qquad y_s = f(x;\, u_1, \cdots, y_{s-1}),$$

analytic for the neighborhood of $\xi[\eta]$ and equal to ζ at ξ $[\eta]$, which makes $A = 0$ for the neighborhood of $\xi[\eta]$.

Let functions $u_1, \cdots, u_q$, analytic at ξ, be constructed which have for themselves, and for their derivatives present in A, at ξ, the corresponding values in $[\eta]$. Let (2) be considered as a differential equation for y, and let $y, \cdots, y_{s-1}$ be given, at ξ, the values which correspond to them in $[\eta]$. Then, by the existence theorem for differential equations, (2) determines y as a function analytic at ξ, and the functions $u_1, \cdots, u_q$; y will constitute a regular solution of A.

21. Let G and H be such that every regular solution of A is a solution of GH. We shall prove that *either every regular solution of A is a solution of G or every regular solution of A is a solution of H.*

Let G_1 and H_1 be, respectively, the remainders of G and H with respect to A. Then, as some $S^p I^t G$ exceeds G_1 by a linear combination of A and its derivatives,* every regular solution of A which annuls G_1 annuls G; similarly for H_1 and H.

If, then, we can show that either G_1 or H_1 is identically zero, our result will be proved. Suppose that neither G_1

* At times we shall, without explicit statement, use symbols, as p and t above, to represent appropriate non-negative integers.

nor H_1 vanishes identically. As G_1 and H_1 are of lower degree than A in y_s, $G_1 H_1 IS$, as a polynomial, is relatively prime to A. Hence, we have

$$B = C(G_1 H_1 IS) + DA,$$

with $B \neq 0$ and free of y_s. As in the discussion of (1), we can build a solution of A for which $G_1 H_1 IS$ does not vanish. But $G_1 H_1$, like GH, vanishes for every regular solution of A. This contradiction proves our result.

22. It follows immediately, from § 21, that the system of all forms which vanish for all regular solutions of A is irreducible. A belongs to this system. The irreducible manifold composed of the solutions of this system will be called the *general solution of* $A = 0$ (or of A).

We show that *every solution of A for which S does not vanish belongs to the general solution.*

Let B be any form which vanishes for all regular solutions. Then some $S^t B$ exceeds, by a linear combination of derivatives of A, a C of order at most s in y. C vanishes for all regular solutions of A. We have

$$I^p C = DA + E, \tag{3}$$

with E reduced with respect to A. Since E vanishes for all regular solutions of A, E, by the discussion of (1), must vanish identically. Thus, as I cannot be divisible by A, C is so divisible. This means that $S^t B$ holds A, so that B vanishes for every solution of A with $S \neq 0$. This proves our statement.

As we shall see later, the general solution may contain solutions with $S = 0$.

Let Σ_1 be the system of all forms which vanish for all solutions of A with $S \neq 0$. In a decomposition of the system A, S into essential irreducible systems, let $\Sigma_2, \cdots, \Sigma_t$ be those systems which are not held by Σ_1. Then

$$\Sigma_1, \Sigma_2, \cdots, \Sigma_t \tag{4}$$

is a decomposition of A into essential irreducible systems.

Thus, *the general solution of A is not contained in any other irreducible manifold of solutions of A. In a decomposition of A into essential irreducible systems, those irreducible systems whose manifolds are not the general solution are held by the separant of A.*

We shall prove that *the general solution of A is independent of the order in which the unknowns in A are taken.*

Suppose that, u_i being some unknown other than y effectively present in A, we order the unknowns so that u_i comes last. With this arrangement, let the manifold of Σ_j in (4) be the general solution of A, and let S' be the separant of A. Suppose that $j \neq 1$. Then S' holds Σ_1, while S holds $\Sigma_2, \cdots, \Sigma_t$. Thus SS' holds A. As was seen in the discussion of (1), this cannot be, since neither S nor S' is divisible by A. This proves our statement.

In Chapter VI, we shall secure a characterization of the solutions of A with $S = 0$ which belong to the general solution. For the present, we limit ourselves to the statement that any solution of A towards which a sequence of solutions with $S \neq 0$ converges uniformly in some area, belongs to the general solution. In short, any form which vanishes for all solutions with $S \neq 0$ will vanish for the given solution.

We can now see that, in the examples in § 15, the systems Σ_2 are irreducible. In each case, the separant vanishes only for $y = 0$, and $y = 0$ gives no solution of Σ_2. Thus, in each case, the manifold of Σ_2 is the general solution.

Closed systems

23. A system Σ will be said to be *closed* if every form which holds Σ is contained in Σ.* Given any system Φ, the system Σ of all forms which hold Φ is closed, and has the same manifold as Φ. Hence no generality will be lost, in the study of manifolds, if we deal only with closed systems.

The only closed system devoid of solutions is the totality of forms with coefficients in $\mathfrak{F}$.

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* A given form is supposed here to occur only once in Σ.

A system which contains non-zero forms, and possesses solutions, will be called *non-trivial.*

Let Σ be a non-trivial closed system in $y_1, \cdots, y_n$. Let

$$(5) \qquad A_1, A_2, \cdots, A_r$$

be a basic set of Σ. Then A_1 is of class greater than 0.

A solution of any ascending set which does not cause the separant or initial of any form of the set to vanish, will be called *a regular solution* of the ascending set.

We shall prove that *every regular solution of* (5) *is a solution of* Σ.

Let S_i and I_i be respectively the separant and initial of A_i.

Let G be any form of Σ. Then the remainder of G with respect to (5) is a form of Σ. This remainder, reduced with respect to (5), must be zero (§ 4). That is, some $S_1^{s_1} \cdots I_r^{t_r} G$ is a linear combination of the A_i and their derivatives. Then G vanishes for every regular solution of (5). Q. E. D.

Suppose now that Σ is irreducible. As no S_i or I_i holds Σ, the product of the S_i and I_i does not hold Σ. It follows, that, *if Σ is irreducible,* (5) *has regular solutions.**

Furthermore, *if Σ is irreducible, any form which vanishes for all regular solutions of* (5) *belongs to* Σ. For, if G is such a form, $S_1 \cdots I_r G$ holds Σ so that G holds Σ.

Thus, *if Σ is irreducible, then Σ is the only closed irreducible system for which* (5) *is a basic set.*

Arbitrary unknowns

24. Let Σ be a non-trivial closed system in $y_1, \cdots, y_n$. There may be some y, say y_j, such that no non-zero form of Σ involves only y_j; that is, every form in which y_j appears effectively also involves effectively some y_k with $k \neq j$. If there exist such unknowns y_j, let us pick one of them, arbitrarily, and call it u_1.

There may be a y, distinct from u_1, such that no non-zero

* In Chapter VI, we determine which solutions of (5) other than the regular ones are solutions of Σ.

form of Σ involves only u_1 and the new y. If there exist such y, let us pick one of them, arbitrarily, and call it u_2.

Continuing, we find a set $u_1, \cdots, u_q\,(q < n)$, such that no non-zero form of Σ involves the u_i alone and such that given any unknown y_j, not among the u_i, there is a non-zero form of Σ in y_j and the u_i alone.

Let the unknowns distinct from the u_i, taken in any order, be represented now by $y_1, \cdots, y_p$, $(p + q = n)$.*

We now list the unknowns† in the order

$$u_1, \cdots, u_q; \qquad y_1, \cdots, y_p. \tag{6}$$

We shall speak generally as if u_i exist. It will be easy to see, in every case, what slight changes of language are necessary when there are no u_i.

Of the non-zero forms in Σ involving only y_1 and the u_i, let A_1 be one of least rank. There certainly exist forms of Σ of class $q+2$ which are reduced with respect to y_1; for instance any non-zero form in y_2 and the u_i alone is of this type. Of such forms, let A_2 be one of least rank.

Continuing, we build a basic set of Σ,

$$A_1, A_2, \cdots, A_p. \tag{7}$$

We shall say that A_i *introduces* y_i.

We shall call $u_1, \cdots, u_q$ a *set of arbitrary unknowns.*

THE RESOLVENT

25. In this section, we assume that $\mathfrak{F}$ does not consist purely of constants.

Let Σ be a non-trivial closed system. Let the unknowns be $u_1, \cdots, u_q$; $y_1, \cdots, y_p$, with the u_i arbitrary unknowns.

We are going to show *the existence in $\mathfrak{F}$ of functions*

$$\mu_1, \cdots, \mu_p \tag{8}$$

* It will be seen in § 30 that when Σ is irreducible, q does not depend on the particular manner in which the u_i may be selected.

† See remarks on notation, §§ 2 and 5.

and the existence of a non-zero form G, free of the y_i, such that either

(a) *There exist no two solutions with the same area of analyticity, and with the same u_i,*

$$
(9) \qquad \begin{array}{ll} u_1, \cdots, u_q; & y_1', \cdots, y_p', \\ u_1, \cdots, u_q; & y_1'', \cdots, y_p'' \end{array}
$$

for which G does not vanish and in which, for some i, y_i' is not identical with y_i'', or

(b) *such pairs of solutions exist, and for each pair,*

$$
(10) \qquad \mu_1 (y_1' - y_1'') + \cdots + \mu_p (y_p' - y_p'')
$$

*is not zero.**

We consider the system of forms obtained from Σ by replacing each y_i by a new unknown z_i. We take the system Ω composed of the forms of Σ, the forms in the z_i just described, and also the form

$$
\lambda_1 (y_1 - z_1) + \cdots + \lambda_p (y_p - z_p),
$$

in which the λ_i are unknowns. That is, Ω involves $3p + q$ unknowns, namely the u_i, y_i, z_i, λ_i.

Let A be any closed essential irreducible system which Ω holds. Suppose that one of the forms $y_i - z_i$, $i = 1, \cdots, p$, does not hold A. We shall prove that A contains a non-zero form which involves no unknowns other than the u_i and λ_i.

If A contains a form in the u_i alone, we have our result. Suppose that A contains no such form.

Since A has all forms in Σ, A has, for $j = 1, \cdots, p$, a non-zero form B_j in y_j and the u_i alone. Let B_j be taken so as to be of as low a rank as possible in y_j. Then S_j, the separant of B_j, is not in A.

Similarly let C_j, $j = 1, \cdots, p$, be a non-zero form of A in z_j and the u_i alone, of as low a rank as possible in z_j. Letting z_j follow the u_i in C_j, we see that the separant S_j' of C_j is not in A.

* If no u_i exist, this is to mean that if Σ has a pair of distinct solutions, (10) does not vanish for the pair. We take $G = 1$ in this case.

To fix our ideas, suppose that $y_1 - z_1$ is not in $\mathcal{A}$. Consider any solution of $\mathcal{A}$ for which

$$(y_1 - z_1) S_1 \cdots S_p S_1' \cdots S_p',$$

(which is not in $\mathcal{A}$) does not vanish. For such a solution, we have

$$(11) \qquad \lambda_1 = -\frac{\lambda_2 (y_2 - z_2) + \cdots + \lambda_p (y_p - z_p)}{y_1 - z_1}. \; .$$

From (11) we find, the jth derivative of λ_1, an expression

$$(12) \qquad \lambda_{1j} = \varrho_j (\lambda_2, \cdots, \lambda_p; y_1, \cdots, y_p; z_1, \cdots, z_p),$$

in which ϱ_j is rational in the λ_i, y_i, z_i and their derivatives, with coefficients in $\mathfrak{F}$. The denominator in each ϱ_j is a power of $y_1 - z_1$.

Let each B_i be of order r_i in y_i and each C_i be of order s_i in z_i.

If a ϱ_j involves a derivative of y_i of order higher than r_i, we can get rid of that derivative by using its expression in the derivatives of y_i of order r_i or less, found from $B_i = 0$. Similarly, we transform each ϱ_j so as to be of order not exceeding s_i in z_i, $i = 1, \cdots, p$.

The new expression for each ϱ_j, which will involve the u_i, will have a denominator which is a product of powers of $y_1 - z_1$; S_i, S_i', $i = 1, \cdots, p$. Let g be the maximum of the integers r_i, s_i. Let

$$h = 2p(g+1)+1.$$

Let k be the total number of letters y_{ij}, z_{ij} which appear in the relations (12), transformed as indicated. Then $h > k$.

We consider the first h of the relations (12).* (That is, we let $j = 0, 1, \cdots, h-1$). Let D, an appropriate product of powers of $y_1 - z_1$, the S_i, S_i', be a common denominator for the second members of these relations. We write

$$(13) \qquad \lambda_{1j} = \frac{E_j}{D},$$

* When $j = 0$, (12) is (11).

$j = 0, \cdots, h-1$. Let D and the E_j, be written as polynomials in the k letters y_{ij}, z_{ij} present in them, with coefficients which are forms in $\lambda_2, \cdots, \lambda_p$ and the u_i. Let m be the maximum of the degrees of these polynomials (total degrees in the y_{ij}, z_{ij}).

Let α represent a positive integer to be fixed later. The number of distinct power products of degree $m\alpha$ or less, in k letters, is*

$$\frac{(m\alpha+k)\cdots(m\alpha+1)}{k!}. \tag{14}$$

Using (13), let us form expressions for all power products of the λ_{1j} in (13) of degree α or less. Let each expression be written in the form

$$\frac{F}{D^\alpha}. \tag{15}$$

Then F, as a polynomial in the y_{ij}, z_{ij}, will be of degree at most $m\alpha$.

The number of power products of the h letters λ_{1j}, of degree α or less, is

$$\frac{(\alpha+h)\cdots(\alpha+1)}{h!}. \tag{16}$$

Now (14) is a polynomial of degree k in α, whereas (16) is of degree h in α. As $h>k$ and as m, h, k are fixed, (16) will exceed (14) if α is large. Let α be taken large enough for this to be realized.

If now the F in (15) are considered as linear expressions in the power products in the y_{ij}, z_{ij}, we will have more linear expressions than power products. Hence the linear expressions F are linearly dependent. That is, some linear combination of the F, with coefficients which are forms in $\lambda_2 \cdots \lambda_p$ and the u_i, not all zero, vanishes identically.

The same linear combination of the power products of the λ_{1j} will vanish for the solution of A for which (11) was written. Now, this last linear combination is a form H

* Perron, *Algebra*, vol. 1, p. 46.

in the u_i and λ_i which is not identically zero, since the power products of the λ_{1j} in H are distinct from one another.

Thus
$$H(y_1-z_1)\,S_1\cdots S_p\,S'_1\cdots S'_p$$
is in $\mathcal{A}$, so that H is in $\mathcal{A}$. This proves our statement.

Let $\mathcal{A}_1, \cdots, \mathcal{A}_r$ be a decomposition of Ω into closed essential irreducible systems. Let $\mathcal{A}_1, \cdots, \mathcal{A}_s$ each not contain some form y_i-z_i and let $\mathcal{A}_{s+1}, \cdots, \mathcal{A}_r$ each contain every y_i-z_i. Let H_i be a non-zero form in $\mathcal{A}_i$, $i=1, \cdots, s$, involving only the u_i and λ_i. Let $K = H_1 \cdots H_s$.

We wish to show the existence in $\mathfrak{F}$ of p functions $\mu_1, \cdots, \mu_p$ such that, when each λ_i is replaced by μ_i in K, then K does not vanish identically in the u_i.

Let K be written as a polynomial in the u_{ij}, with forms in the λ_i as coefficients. Let L be one of the coefficients in K. If we can fix each λ_i in $\mathfrak{F}$ so that L does not vanish, our result will be established.

Let ζ be any non-constant function in $\mathfrak{F}$, and let a be a point of $\mathfrak{A}$ at which ζ is analytic and has a non-vanishing derivative. Given a sufficiently small circle with a as center, any function φ, analytic in the circle, can be expressed as a power series in ζ with constant coefficients. Then φ can be approximated uniformly within the circle by a polynomial in ζ. Thus if m is a sufficiently large integer, and if $t_{i0}, \cdots, t_{im}$, $i=1, \cdots, p$ are arbitrary constants, L cannot vanish identically in the t_{ij} if each λ_i is replaced in L by
$$t_{i0}+t_{i1}\zeta+\cdots+t_{im}\zeta^m.$$
Otherwise L would vanish if the λ_i are any functions analytic in the above circle. Thus there must be integral values of the t_{ij} for which L does not vanish. Every polynomial in ζ with integral coefficients is in $\mathfrak{F}$. This shows the existence of the required μ_i.

The solutions of Ω for $\lambda_j=\mu_j$, $j=1, \cdots, p$, will be the solutions of the $\mathcal{A}_i$ for $\lambda_j=\mu_j$. Now, the solutions with $\lambda_j=\mu_j$ of $\mathcal{A}_1, \cdots, \mathcal{A}_s$ have u_i which cause to vanish the

form G obtained by putting $\lambda_j = \mu_j$ in K. The solutions of $A_{s+1}, \cdots, A_r$, even with $\lambda_j = \mu_j$, have $y_i = z_i$, $i = 1, \cdots, p$.

When every A_j contains every $y_i - z_i$, we take $G = 1$, $\mu_1 = \cdots = \mu_p = 0$.

We have thus the result stated at the head of this section.*

26. We shall now relinquish the condition that $\mathfrak{F}$ contain a non-constant function

Let us assume that u_i exist. We are going to prove *the existence of forms G, $M_1, \cdots, M_p$, in the u_i alone, with $G \neq 0$, such that, for two distinct solutions* (9) *for which G does not vanish,*

(17) $$M_1(y_1' - y_1'') + \cdots + M_p(y_p' - y_p'')$$

is not zero.

The discussion of § 25 holds through the construction of the form K. We are going to prove the existence of forms $M_1, \cdots, M_p$ in the u_i alone, such that, when λ_i is replaced by M_i in K, the resulting form G is not identically zero.

Let K be arranged as a polynomial in the λ_{1j}, with forms in the u_i, $\lambda_2, \cdots, \lambda_p$ as coefficients. Let u_{1h} be the highest derivative of u_1 which appears in any of the coefficients. Let k be an integer greater than h. Then, if λ_1 is replaced by u_{1k}, K becomes a form K_1 in the u_i and $\lambda_2, \cdots, \lambda_p$, which is not identically zero. Similarly, if we replace λ_2 in K_1 by a sufficiently high derivative of u_1, we obtain a non-zero form K_2 in the u_i and $\lambda_3, \cdots, \lambda_p$. Replacing $\lambda_3, \cdots, \lambda_p$ in succession by sufficiently high derivatives of u_1, we obtain a non-zero form G.

Continuing as in § 25, we see that the solutions of Ω in which $\lambda_j = M_j$, $j = 1, \cdots, p$ are the solutions of the A_i which satisfy $\lambda_j = M_j$. Now the solutions with $\lambda_j = M_j$ of $A_1, \cdots, A_s$ have u_i which cause G to vanish. The so-

---

* The following example shows that Σ may have many solutions with given u_i, and that a G may exist, such that, for $G \neq 0$, there is only one solution for given u_i. Let the unknowns be u_1, u_2, y_1. Let Σ consist of all forms which hold $u_1 y_1 - u_2$. Let $G = u_1$. Then u_1, u_2 is a set of arbitrary unknowns. If $u_1 = u_2 = 0$, y_1 may be taken arbitrarily, but, for given u_1, u_2 with $G \neq 0$, there is only one y_1.

lutions of $A_{s+1}, \cdots, A_r$, even with $\lambda_j = M_j$, have $y_i = z_i$, $i = 1, \cdots, p$. This proves our statement.

27. The results of §§ 25, 26 permit us to state that if either

(a) $\mathfrak{F}$ does not consist purely of constants or

(b) there exist u_i,

then triads of forms G, P, Q, exist with G and P not in Σ, and G free of the y_i, such that, for any two distinct solutions of Σ, with the same u_i, such that neither G nor P vanishes, the expression Q/P yields two distinct functions of x. For instance, if (a) holds, we can take $P = 1$ and $Q = \mu_1 y_1 + \cdots + \mu_p y_p$.

It will essentially increase the generality of our work to use general forms P. The following is a non-trivial example in which P is of class greater than 0. Let $\mathfrak{F}$ be the totality of rational functions of x. Let the unknowns be y_1, y_2 and let Σ consist of all forms which hold y_{11} und y_{21}. The solutions are $y_1 = c$, $y_2 = d$, with c and d constant but arbitrary. We take $G = 1$. If

$$P = y_1 + x y_2,$$
$$Q = y_1^2 + x^2,$$

the expression Q/P gives distinct functions of x for distinct solutions of Σ with $P \neq 0$.

In certain cases in which $\mathfrak{F}$ consists purely of constants and in which no u_i exist, there may exist no pair P, Q as described above. For instance, let $\mathfrak{F}$ be the totality of constants. Let the unknowns and Σ be as in the preceding example. The y_{ij} are all zero for $j > 0$ for every solution. We therefore lose no generality in seeking a P and Q of order zero in y_1, y_2. For any such P and Q, Q/P will yield the same result, for infinitely many distinct pairs of constants y_1, y_2.

In developing the theory of an irreducible system Σ for the case in which $\mathfrak{F}$ has only constants and there are no u_i, two courses are open to us. If we adjoin x to $\mathfrak{F}$, then,

by § 18, Σ will remain irreducible in the enlarged field. Working in the enlarged field, we can secure a P and Q. Again, by § 17, we can introduce a new unknown u_1 and Σ will remain an irreducible system. After either type of adjunction, the theory which follows will apply.

28. In §§ 28, 29 we deal with a non-trivial closed irreducible system Σ. We assume that either

(a) $\mathfrak{F}$ does not consist entirely of constants, or

(b) arbitrary unknowns exist.

We take a triad G, P, Q, as in § 27.

We introduce a new unknown, w, and consider the system Λ obtained by adjoining the form $Pw-Q$ to Σ. Let Ω be the system of all forms in w, the u_i and y_i which vanish for all solutions of Λ with $P \neq 0$.* We shall prove that Ω is irreducible.

Let B and C be such that BC holds Ω. For s appropriate, $P^s B$ minus a linear combination of $Pw-Q$ and its derivatives, is a form R free of w. We obtain similarly, from a $P^t C$, an S, free of w. Then RS vanishes for every solution of Σ with $P \neq 0$, since every such solution yields a solution of Ω. Hence PRS holds Σ, so that either R or S is in Σ. If R is in Σ, $P^s B$ holds Ω. Hence B vanishes for all solutions of Λ with $P \neq 0$, so that B is in Ω. Thus Ω is irreducible.

We notice that those forms of Ω which are free of w are precisely the forms of Σ. In particular, Ω contains no non-zero form in the u_i alone.

We are going to show that Ω contains a non-zero form in w and the u_i alone.

Let B_i, $i=1,\cdots,p$, be a non-zero form of Σ involving only y_i; $u_1,\cdots,u_q$, of minimum rank in y_i. Let S_i be the separant of B_i.

Consider any solution of Ω for which $PS_1\cdots S_p$ does not vanish. For such a solution, we have

$$w=\frac{Q}{P}.$$

* Of course, forms in Ω may also vanish when $P=0$.

For the jth derivative of w, we have an expression

$$w_j = \frac{Q_j}{P^{j+1}}. \tag{18}$$

Using the relations $B_i = 0$, we free each Q_j from derivatives of each y_i of order higher than the maximum of the orders of Q, P and B_i in y_i. Each w_j will then be expressed as a quotient of two forms, the denominator being a product of powers of $P, S_1, \cdots, S_p$. If we use a sufficient number of the relations (18), as just transformed, we will have more w_j than there are y_{ij} in the second members. Using the process of elimination employed in § 25, we obtain a non-zero form K in $w; u_1, \cdots, u_q$ which vanishes for every solution of Ω with $PS_1 \cdots S_p \neq 0$. As $PS_1 \cdots S_p$ is not in Ω, and as Ω is irreducible, K is in Ω.

29. We now list the unknowns in Ω in the order

$$u_1, \cdots, u_q; \quad w; \quad y_1, \cdots, y_p$$

and take a basic set for Ω,

$$A, A_1, \cdots, A_p. \tag{19}$$

Here, $w, y_1, \cdots, y_p$ are introduced in succession. (See final remarks of § 24.)

If A is not algebraically irreducible, we can evidently replace it by some one of its irreducible factors. We assume, therefore, that *A is algebraically irreducible.*

We are going to prove that $A_1, \cdots, A_p$ are of order 0 in $y_1, \cdots, y_p$, and, indeed, that A_i is of the first degree in y_i. Thus, since A_i with $i > 1$ will be of lower degree in y_j than A_j with $j < i$, *each equation $A_i = 0$ expresses y_i rationally in terms of $w; u_1, \cdots, u_q$ and their derivatives.*

The determination of the manifold of Σ will in this way be made to depend on the determination of the general solution of $A = 0$, which equation will be called a *resolvent* of Σ.*

* If A is any system equivalent to Σ, we also call $A = 0$ a resolvent of A.

Suppose that A_1 is of order higher than zero in y_1. Consider any regular solution of (19) for which PG does not vanish. By the final remarks of § 23, such regular solutions exist. Let ξ be any point at which the functions in this solution and the coefficients in

$$P,\ G,\ A,\ A_1,\ \cdots,\ A_p$$

are analytic, and for which, if S and I are the separant and initial of A, S_i and I_i those of A_i,

$$PGSS_1\ \cdots\ S_p II_1\ \cdots\ I_p \neq 0.$$

Without changing w or the u_i in the solution, we can alter slightly the values at ξ of the derivatives of y_1, in A_1, other than the highest, and obtain a second regular solution of (19) with $PG \neq 0$. That is, we can solve $A_1 = 0$ for y_1 with the modified initial conditions, substitute the resulting y_1 into A_2, solve $A_2 = 0$ with the same initial conditions for y_2 which obtained in the first regular solution,* and, continuing, determine each y_i. This is nothing but an application of the implicit function theorem, and of the existence theorem for differential equations. Thus, we would have two distinct solutions of Ω, with the same u_i, with $PG \neq 0$, and with the same w. This contradicts the fundamental property of the triad G, P, Q.

Hence, A_1 is of order zero in y_1. Similarly, every A_i is of order zero in y_i. Furthermore, as A_i is of lower rank in y_j than A_j for $j < i$, each A_i is of zero order in y_j for $j \leqq i$.

We shall now prove that each A_i is linear in y_i.

We start with A_p. Suppose that A_p is not linear in y_p. Let P_1 be the remainder for P with respect to (19). Then every regular solution of (19) which causes either of the forms P, P_1 to vanish, causes the other to vanish.

---

* That is, with the same values at ξ for y_2 and all its derivatives but the highest.

If we can show that the ascending set

$$A,\ A_1,\ \cdots,\ A_{p-1} \tag{20}$$

has a regular solution

$$u_1,\ \cdots,\ u_q;\quad w;\quad y_1,\ \cdots,\ y_{p-1}$$

for which A_p has two distinct solutions in y_p with $S_p I_p P_1 G \neq 0$, we shall have forced a contradiction.

If we cannot get two distinct solutions of this type, it must be that for every regular solution of (20) with $I_p \neq 0$, the equation $A_p = 0$ has a solution in y_p for which $S_p P_1 G$ vanishes.*

Let C be a remainder for $S_p P_1 G$ with respect to A_p considered as an ascending set. Then C is of zero order in every y_i and is of lower degree than A_p in y_p. Every common solution of $S_p P_1 G$ and A_p is a solution of C. We note that C, like S_p, P_1 and A_p, is not of higher order in w than A.

Of all forms not in Ω, of zero order in the y_i, which are of lower degree than A_p in y_p, not of higher order in w than A, and which, for every regular solution of (20) with $I_p \neq 0$ have an annulling function y_p in common with A_p, let D be one which has a minimum degree in y_p. Then D must be at least of the first degree in y_p, else $I_p D$ would vanish for every regular solution of (19) and would be in Ω.

Let K be the initial of D. Then K is not in Ω, else D would not be of a minimum degree in y_p. For m appropriate,

$$K^m A_p = ED + F,$$

with E of lower degree than A_p in y_p and F of lower degree than D in y_p. Every annulling function y_p of A_p and D makes F vanish. Then F must be in Ω.

Thus ED must be in Ω, so that E, which is not zero, is in Ω. Let

$$E = H_0 + H_1 y_p + \cdots + H_t y_p^t \tag{21}$$

* Because (19) has regular solutions, (20) has regular solutions with $I_p \neq 0$.

with the H_i forms free of y_p and of order in w not greater than that of A. We understand that $H_t \neq 0$.

As $K^m A_p$ is of higher degree in y_p than F, the initial of ED is identical with that of $K^m A_p$ and hence is not in Ω. Then H_t is not in Ω.

Evidently a non-negative integer a_1 exists such that, when a suitable multiple of A_{p-1} is subtracted from $I_{p-1}^{a_1} H_i$, $i = 0, \cdots, t$, the remainder is of lower degree than A_{p-1} in y_{p-1}.* In the same way, we find integers $a_2, \cdots, a_{p-1}$; a such that when a suitable linear combination of the forms of (20) is subtracted from

$$I_{p-1}^{a_1} \cdots I_1^{a_{p-1}} I^a H_i,$$

$i = 0, \cdots, t$, the remainder is reduced with respect to (20). As

$$I_{p-1}^{a_1} \cdots I^a E$$

is in Ω, we see that Ω contains a form

$$E_1 = H_0' + \cdots + H_t' \, y_p^t,$$

with each H_i' reduced with respect to (20) and with H_t' not in Ω (hence not 0). As t is less than the degree of A_p in y_p, E_1, which is not zero, is reduced with respect to (19).

This contradiction (§ 4), proves that A_p is linear in y_p.

We now consider A_{p-1}, assuming that it is not linear in y_{p-1}. Since P_1 is of lower degree in y_p than A_p, P_1 is free of y_p. It must be that, for every regular solution of

$$(22) \qquad A, A_1, \cdots, A_{p-2}$$

with $I_{p-1} \neq 0$, $A_{p-1} = 0$ has a solution in y_{p-1} for which $S_{p-1} I_p P_1 G$ vanishes. The proof continues as for A_p.

In dealing with A_{p-2}, we consider that both P_1 and I_p are free of y_{p-1}. The proof continues as above.

Thus every A_i is linear in y_i, and *each y_i has an expression rational in w; $u_1, \cdots, u_q$ and their derivatives, with coefficients in $\mathfrak{F}$.*

---

* Note that I_{p-1} is free of y_p.

We notice that, *if*

$$u_1, \cdots, u_q; w; y_1, \cdots, y_p$$

*is a solution of Ω, then $u_1, \cdots, u_q; w$ belongs to the general solution of A.**

For, if a form K in w and the u_i vanishes for every solution in the general solution of A, then K vanishes for every regular solution of (19), and so is in Ω.

When Q is in Σ, $w = 0$ is a resolvent. Then each y_i is rational in the u_{ij}.

The introduction of the resolvent accomplishes the following:

(a) It reduces the study of an irreducible system to the study of the general solution of a single equation. Of course for solutions of Σ with $P = 0$, there may be no corresponding w and, for other solutions of Σ, the initial of some A_i may vanish. We shall gain information as to these exceptional solutions in Chapter VI.

(b) It leads to a theoretical process for constructing all irreducible systems (§ 32).

(c) It creates an analogy between $y_1, \cdots, y_p$ and a system of p algebraic functions of q variables. It is well known, in short, that, given such a system of algebraic functions, we can find a single algebraic function in terms of which, and of the variables, the functions of the system can be expressed rationally.

(d) It furnishes an instrument useful in the solution of formal problems.

INVARIANCE OF THE INTEGER q

30. We consider a non-trivial closed irreducible system Σ in any field $\mathfrak{F}$.

We propose to show that, if arbitrary unknowns exist, the number q of arbitrary unknowns does not depend on the manner in which the u_i are selected.

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* Here we consider A as a form in w and the u_i alone.

Let u_i exist. It will suffice to prove that, given any $q+1$ unknowns among the u_i and y_i,

$$z_1, \cdots, z_{q+1},$$

there exists a non-zero form in Σ which involves only the z_i.

We form a resolvent for Σ. As u_i exist, this is possible. Let us consider the regular solutions of (19). Every z_i in such a solution has a rational expression in $u_1, \cdots, u_q; w$. If a z_i happens to be a u, say u_j, the expression for that z_i is simply u_j. We write

$$(23) \qquad z_i = \varrho_i\,(w;\, u_1, \cdots, u_q),\ (i = 1, \cdots, q+1).$$

On differentiating (23) repeatedly, we get expressions for the z_{ij} which are rational in the w_j and u_{ij}. Making use of the relation $A = 0$, we transform these relations so as not to contain derivatives of w of order higher than r, where r is the order of A in w.

None of the expressions thus obtained will have a denominator which vanishes for a regular solution of (19).

Since there are $q+1$ of the z_i, and only q of the u_i, it follows that if we differentiate (23) often enough (and then transform), the z_{ij} will become more numerous than the u_{ij} and $w, w_1, \cdots, w_r$.

It follows, as in § 25, that there exists a non-zero form in the z_i which vanishes for all regular solutions of (19). The form thus obtained belongs to Σ. The invariance of q is proved.

The assumption that Σ is irreducible is essential. For instance, consider the system

$$u_1\,y_1 = u_2\,y_2 = u_3\,y_2 = 0,$$

in the unknowns $u_1, u_2, u_3; y_1, y_2$. These equations impose no relations either upon the y_i or upon the u_i. Thus u_1, u_2, u_3 and y_1, y_2 are two sets of arbitrary unknowns.

ORDER OF THE RESOLVENT

31. We work with any non-trivial closed irreducible system Σ for which triads G, P, Q, and therefore resolvents, exist. Considering Σ as a system in the u_i and y_i, let

$$A_1, \cdots, A_p \tag{24}$$

be a basic set for Σ, the separant and initial of A_i being S_i and I_i respectively.

Let the order of A_i in y_i be r_i. Let

$$h = r_1 + \cdots + r_p.$$

We shall prove that *every resolvent of* Σ *is of order* h *in* w.

We begin by proving that Ω contains a non-zero form in w; $u_1, \cdots, u_q$ whose order in w does not exceed h.

Consider any solution of Ω for which

$$PS_1 \cdots S_p I_1 \cdots I_p \neq 0. \tag{25}$$

For such a solution, we have

$$w = \frac{Q}{P}. \tag{26}$$

We propose to show the existence of forms R and T each of order not exceeding r_i in y_i, $i = 1, \cdots, p$, such that, for any solution of Ω for which (25) holds, T is not zero and

$$w = \frac{R}{T}. \tag{27}$$

Let Q_1 and P_1 be the remainders of Q and P respectively, relative to (24). Let Q_1 be obtained by subtracting a linear combination of the A_i and their derivatives from

$$S_1^{s_1} \cdots I_p^{t_p} Q$$

and let P_1 be obtained similarly from

$$S_1^{\sigma_1} \cdots I_p^{\tau_p} P.$$

Then, if (25) holds, we have

$$w = \frac{Q_1 S_1^{\sigma_1} \cdots I_p^{\tau_p}}{P_1 S_1^{s_1} \cdots I_p^{t_p}}. \tag{28}$$

For R and T in (27) we take the numerator and denominator in (28) respectively.

We find, from (27), for the jth derivative of w, an expression

$$w_j = \frac{B_j}{T^{j+1}}. \tag{29}$$

If U_j is the remainder of B_j with respect to (24), we can write (29)

$$w_j = \frac{U_j}{W_j}, \tag{30}$$

where W_j is a product of powers of $T, S_1 \cdots I_p$.

Consider (27) and the first h relations (30). Let D be a common denominator for the second members in these $h+1$ relations. We write

$$w_j = \frac{E_j}{D}, \tag{31}$$

$j = 0, \cdots, h$.

Let D, the E_j and the A_i be written as polynomials in the y_{ij} with coefficients which are forms in the u_i. Let m be the maximum of the degrees of these polynomials.

For convenience, we represent the r_ith derivative of y_i by z_i. Let A_i be of degree v_i in z_i.

Let α be a positive integer, to be fixed later. In (31), let us form all power products in the w_j of degree α or less. Let the expression for each power product be written in the form

$$\frac{F}{D^\alpha}. \tag{32}$$

Then each F is a polynomial in the y_{ij}, of degree not exceeding $m\alpha$.

Let each expression (32) be written

$$\frac{F I_p^{m\alpha}}{D^\alpha I_p^{m\alpha}}. \tag{33}$$

Consider a particular F, and let it be written as a polynomial in z_p. Suppose that its degree d in z_p is not less than m. Then, as $A_p = 0$ for the solution of Ω which we are considering, we have, letting

$$M = A_p - I_p z_p^{v_p},$$

the relation

(34) $$I_p z_p^d = - M z_p^{d-v_p}.$$

If

$$F = J_0 + J_1 z_p + \cdots + J_d z_p^d,$$

with the J_i free of z_p, we may write the numerator in (33) in the form

(35) $$(J_0 I_p + \cdots + J_d I_p z_p^d)\, I_p^{m\alpha-1}.$$

Since I_p is of degree less than m in the y_{ij}, each term in the parenthesis in (35) is of degree less than $m(\alpha+1)$.

We replace $J_d I_p z_p^d$ by $-J_d M z_p^{d-v_p}$ in (35). As J_d is of degree not exceeding $m\alpha - d$ in the y_{ij} and as M is of degree at most m, then $J_d M z_p^{d-v_p}$ is of degree less than $m(\alpha+1)$ in the y_{ij}. Thus, (33) goes over into

$$\frac{F_1 I_p^{m\alpha-1}}{D^\alpha I_p^{m\alpha}},$$

where F_1 is of degree less than $m(\alpha+1)$ in the y_{ij} and of degree less than d in z_p. If the degree of F_1 in z_p is not less than m, we repeat the above operation. After $t \leqq m\alpha$ operations, we get an expression

(36) $$\frac{H I_p^{m\alpha-t}}{D^\alpha I_p^{m\alpha}}$$

with H of degree less than m in z_p and of degree less than $m(\alpha+t)$ in the y_{ij}. The numerator in (36) is of degree in the y_{ij} less than

$$m(\alpha+t) + m(m\alpha - t) \leqq 2m^2\alpha.$$

Thus, if we let $D_1 \doteq D I_p^m$, we can write each power product in the w_j, of degree α or less, in the form

(37) $$\frac{K}{D_1^\alpha},$$

where K is of degree less than $2m^2\alpha$ in the y_{ij} and of degree less than m in z_p.

We now write each expression (37) in the form

$$\frac{K\, I_{p-1}^{2m^2\alpha}}{D_1^\alpha\, I_{p-1}^{2m^2\alpha}}. \tag{38}$$

and employ, with respect to z_{p-1}, the procedure used above. We find for each expression (38), an equivalent expression

$$\frac{L}{D_2^\alpha} \tag{39}$$

with $D_2 = D_1\, I_{p-1}^{2m^2}$ and with L of degree less than $4m^3\alpha$ in the y_{ij} and of degree less than m in z_p and z_{p-1}. Continuing, we find an expression for each power product of the w_j

$$\frac{W}{D_p^\alpha} \tag{40}$$

where W is of degree less than $2^p\, m^{p+1}\,\alpha$ in the y_{ij}, and of degree less than m in z_i, $i = 1, \cdots, p$. Let c represent $2^p\, m^{p+1}$.

The number of power products in $z_1, \cdots, z_p$, of degree less than m in each letter, is m^p.

Hence the number of power products of the y_{ij} of degree $c\alpha$ or less, and of degree less than m in each z_i, is not more than

$$m^p\,\frac{(c\alpha + h)\ \cdots\ (c\alpha + 1)}{h!}. \tag{41}$$

This is because the y_{ij} with $j < r_i$ are h in number. On the other hand, the number of power products of degree α or less in the $h+1$ w_j is

$$\frac{(\alpha + h + 1) \cdots (\alpha + 1)}{(h+1)!}. \tag{42}$$

As (42) is of degree $h+1$ in α and (41) only of degree h, (42) will exceed (41) for α large. This, as we know from § 25, implies the existence of a non-zero form of Ω in w and the u_i alone, of order not exceeding h in w.

This shows that the order in w of the resolvent $A=0$ does not exceed h. Suppose that the order of A is $k<h$. For each y_i, we have an expression

$$y_i = \frac{C_i}{D_i} \tag{43}$$

with C_i and D_i forms in w; $u_1, \cdots, u_q$, of order not exceeding k in w. We obtain from (43) expressions for the y_{ij}, $j=0, \cdots, r_i-1$, which are rational in the w_j, u_{ij}, with powers of the D_i as denominators. Using the relation $A=0$, we depress the orders in w of the numerators until they do not exceed k. The transformed expressions will have denominators which are power products of the D_i and S.

By an elimination we obtain a non-zero form W in the y_i, u_i which belongs to Ω, hence to Σ. This W, which is of order less than r_i in each y_i, is reduced with respect to (24). This is impossible.

We have thus proved that the order in w of every resolvent is h.

We say now that, *when $u_1, \cdots, u_q$ are selected, the quantity $r_1 + \cdots + r_p$ does not depend on the manner in which the subscripts $1, \cdots, p$ are assigned to the remaining unknowns.*

This follows immediately from what precedes, except that we have to prove that, when no u_i exist and $\mathfrak{F}$ has only constants, $r_1 + \cdots + r_p$ is independent of the order of the y_i. What we do is to introduce a new unknown, u_1. Σ will remain irreducible, u_1 will be an arbitrary set, and (24) will remain a basic set. The methods above then apply.

The degree of the resolvent in w_h does depend on P and Q. Consider, for instance, the system, irreducible in the field of all rational functions

$$y_{11}-1, \quad y_2-y_1^2.$$

As the manifold is $y_1 = x+a$, $y_2 = (x+a)^2$, we may evidently take $w=y_1$. The resolvent becomes $w_1-1=0$. On the other hand, if we take $w=y_1+y_2$, the resolvent becomes of the second degree in w_1.

The order of the resolvent depends on the choice of the u_i. For instance

$$y_{11} - y_2 = 0$$

is irreducible in the field of all constants. If we let $u_1 = y_2$, we get a resolvent of the first order. If we let $u_1 = y_1$, we get a resolvent of zero order.

Construction of irreducible systems

32. We shall establish a result which is, to some extent, a converse of the result of § 29.

Let A be an algebraically irreducible form in $u_1, \cdots, u_q;$ w, effectively involving w. Let

$$(44) \qquad y_i = \frac{P_i}{Q}, \qquad i = 1, \cdots, p,$$

where the P_i and Q are forms in $u_1, \cdots, u_q;$ w and where Q does not vanish for every solution in the general solution of A. Let $\bar{u}_1, \cdots, \bar{u}_q;$ $\bar{w}$ be any solution in the general solution of A which does not annul Q. For this solution, we obtain, from (44), functions $\bar{y}_1, \cdots, \bar{y}_p$.

Let Ω be the system of forms in the u_i, y_i, w which vanish for all $\bar{u}_i$, $\bar{y}_i$, $\bar{w}$. We shall prove that Ω is irreducible.

Let GH hold Ω. If we substitute (44) into G, we get

$$G = \frac{T}{U},$$

with T a form in $u_1, \cdots, u_q;$ w and U a power of Q. We have, similarly, $H = V/W$. For any $\bar{u}_i$, $\bar{w}$, TV vanishes. Then TVQ vanishes for every solution in the general solution of A. Thus either T vanishes for every such solution of A, or V does. Consequently one of G, H must vanish for all $\bar{u}_i$, $\bar{y}_i$, $\bar{w}$. Hence Ω is irreducible.

By the method of § 25, we can show that Ω contains non-zero forms in the u_i, y_i alone. The system Σ composed of all such forms and the zero form is a closed irreducible system. What is more, the theory of resolvents shows that

every closed irreducible system in $y_1, \cdots, y_n$ can be obtained in this way. We have thus a theoretical process for constructing all closed irreducible systems.

IRREDUCIBILITY AND THE OPEN REGION $\mathfrak{A}$

33. The question might be raised as to whether Σ, irreducible in $\mathfrak{F}$ for the open region $\mathfrak{A}$, can be reducible in $\mathfrak{F}$ for some open region $\mathfrak{A}_1$ in $\mathfrak{A}$. We shall show that the answer is negative.

Without loss of generality, we assume Σ non-trivial and closed. Also, we assume, adjoining x to $\mathfrak{F}$ if necessary, that $\mathfrak{F}$ does not consist purely of constants.

Let a form K vanish for all solutions of Σ which are analytic in a part of $\mathfrak{A}_1$. We shall prove that K vanishes for all solutions of Σ. Suppose that K is not in Σ. We construct a resolvent, and consider (19). Let K_1 be the remainder of K with respect to (19). Then K_1 is not divisible by A. On the other hand,

$$K_1 S I I_1 \cdots I_p \tag{45}$$

(as in § 29), vanishes for every solution of A which is analytic in a part of $\mathfrak{A}_1$. This is impossible, because (45) is not divisible by A (§ 20).

Thus if P and Q are forms such that PQ vanishes for all solutions of Σ analytic in a part of $\mathfrak{A}_1$, then PQ vanishes for all solutions of Σ. This means that either P or Q is in Σ, so that Σ is irreducible in $\mathfrak{A}_1$.

CHAPTER III

FIRST APPLICATIONS OF THE GENERAL THEORY

RESULTANTS OF DIFFERENTIAL FORMS

34. In algebra, in developing the theory of resultants of systems of polynomials, it is necessary to deal with polynomials whose coefficients are indeterminates. So, in connection with resultants of pairs of differential forms, we shall find it desirable to deal with *general* forms. For our purposes, it will be convenient to define a *general form* in y as one of the type

$$A = a_0 + a_1 y + a_2 P_2 + \cdots + a_n P_n,$$

with $n \geqq 1$ where the a_i are *indeterminates* and where the P_i, $i = 0, \cdots, n$, ($P_0 = 1$, $P_1 = y$), are distinct power products in y and its derivatives. By an indeterminate, we mean a symbol which can be replaced, when it is desired, by an arbitrarily assigned analytic function.*

Consider a second general form

$$B = b_0 + b_1 y + b_2 Q_2 + \cdots + b_m Q_m.$$

We propose to find a condition upon the a_i and b_i, necessary for the existence of a common solution in y of A and B.

Let the a_i and b_i be considered now as unknowns, and let A and B be considered as forms in the a_i, b_i and y in any field $\mathfrak{F}$. We shall show that the system

* It is understood that, when the a_i are replaced by analytic functions, the replacing functions have a common domain of analyticity. In defining a general form, we do not use the notion of a coefficient field.

$$(1) \qquad A, B$$

is irreducible.

Let X and Y be forms such that XY holds (1). We have

$$(2) \quad a_0 = -a_1 y - \cdots - a_n P_n; \quad b_0 = -b_1 y - \cdots - b_m Q_m.$$

If a_0 and b_0 are replaced in X and in Y by the second members in (2), there result two forms X_1 and Y_1 in

$$(3) \qquad a_1, \cdots, a_n; \quad b_1, \cdots, b_m; \quad y$$

such that $X_1 Y_1$ holds (1). But as the unknowns (3) may be taken arbitrarily, as analytic functions, and a_0, b_0 be determined by (2) so as to make $A = B = 0$, it must be that $X_1 Y_1$ vanishes identically. Then one of X_1, Y_1 must vanish identically, and one of X, Y must hold (1).

Let Σ be the system of all forms which hold (1). Using (2), it can be shown, by the method of § 25, that Σ contains non-zero forms in the a_i and b_i alone.

On the other hand, Σ contains no non-zero form in

$$(4) \qquad a_1, \cdots, a_n; \quad b_0, \cdots, b_m.$$

Suppose that such a form, C, exists. Let the unknowns in (4) be taken as analytic functions, with $b_1, \cdots, b_m$ not all zero, so that $C \neq 0$. Then certainly $B = 0$ has a solution in y. Using any such solution y, we can determine a_0 so that $A = 0$. C cannot exist.

Suppose now that $\mathfrak{F}$ is the field of all rational constants. Let (4) be the arbitrary unknowns, and let

$$(5) \qquad R, U$$

be a basic set for Σ, R and U introducing a_0 and y respectively.

We assume that R is algebraically irreducible and that its coefficients are relatively prime integers. This determines R uniquely, except for algebraic sign. For, let S be any other form which satisfies the conditions placed on R. Then S and R are of the same rank. The remainder of S with

respect to R, being in Σ, must be zero. Then I being the initial of R, some $I^p S$ is divisible by R. Hence S is divisible by R, and as the coefficients in S are relatively prime, we have $S = \pm R$. We suppose the sign of R to be fixed according to any suitable convention, and treat R as unique.

We shall call R the *resultant* of A and B.

We shall now prove that U is of order 0 in y and, indeed, that U is *linear* in y.

Let $\mathfrak{F}_1$ be the field obtained by adjoining x to $\mathfrak{F}$. We form a resolvent for (1) in $\mathfrak{F}_1$, using a w defined by

$$w = a_0 + \mu y, \tag{6}$$

with μ a rational function of x. Let the resolvent be $V = 0$ and let $y = N/M$, with M and N forms in the a_i, b_i, and w.

The system

$$A, \; B, \; w - a_0 - \mu y \tag{7}$$

is equivalent to the system

$$w + (a_1 - \mu) y + a_2 P_2 + \cdots + a_n P_n; \quad B; \quad a_0 - w + \mu y. \tag{8}$$

A solution of the first two forms in (8) will satisfy $V = 0$, $My - N = 0$. If, then, V_1, M_1, N_1 are the forms which result from V, M, N respectively on replacing w by a_0 and a_1 by $a_1 + \mu$,

$$V_1, \; M_1 y - N_1 \tag{9}$$

will be a basic set in $\mathfrak{F}_1$ for Σ_1, the totality of all forms in $\mathfrak{F}_1$ which hold (1).

Evidently V_1 cannot be of higher rank in a_0 than R. This implies that M_1 and N_1 are of lower rank in a_0 than R.

Let $M_1 y - N_1$ be written in the form

$$\frac{(S_1 y - T_1) + \cdots + (S_r y - T_r) x^r}{\alpha} \tag{10}$$

with the S_i and T_i forms in the a_i and b_i, with integral coefficients and with α a polynomial in x.

The numerator in (10) holds Σ. By § 18, each $S_i y - T_i$ holds Σ. Let j be such that $S_j \neq 0$. Then $S_j y - T_j$ is a non-zero form of Σ reduced with respect to R. This proves that U, in (5), is linear in y.

Thus, *for A and B to have a common solution in y, it is necessary that*

$$a_0, \cdots, a_n;\ b_0, \cdots, b_m$$

be a solution in the general solution of the resultant of A and B. If $a_0, \cdots, b_m$ is such a solution, and if it does not annul a certain fixed form in $a_0, \cdots, b_m$, then A and B have a single solution in common, which can be expressed rationally in terms of $a_0, \cdots, b_m$, with integral coefficients.*†

We prove now that *the resultant of A and B is a linear combination of A, B and a certain number of their derivatives, the coefficients in the linear combination being forms with integral coefficients.*

In R, let a_0 and b_0 be replaced respectively by

$$A - a_1 y - \cdots - a_n P_n, \qquad B - b_1 y - \cdots - b_m Q_m,$$

and let R be expanded as a polynomial in A, B and their derivatives. The term not involving A, B, or their derivatives, will be a form in the unknowns (3) which holds (1). As we saw, such a form vanishes identically. This gives our result.‡

The methods of Chapter V permit the actual construction of resultants.

Analogue of an Algebraic Theorem of Kronecker

35. It is a theorem of Kronecker that, given any system of algebraic equations in n unknowns, there exists an equivalent system containing $n+1$ or fewer equations.§ We present an analogous theorem for differential equations.

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* The coefficient of y in U.

† We are using the expression "single solution" in the sense of analytic function theory rather than in the sense of § 6.

‡ For a theory of resultants of linear differential forms, see Heffter, Journal für die r. u. a. Mathematik, vol. 116 (1896), p. 157.

§ König, *Algebraische Größen*, p. 234.

THEOREM. *Let $\mathfrak{F}$ contain a non-constant function. Let Σ be any system of forms in $y_1, \cdots, y_n$. Then there exists a system Φ, composed of $n+1$ or fewer forms, whose manifold is identical with that of Σ. If Σ consists of a finite number of non-zero forms*

$$F_1, \cdots, F_r, \tag{11}$$

then a system Φ exists which is composed of F_1 and n or fewer linear combinations, with coefficients in $\mathfrak{F}$, of $F_2, \cdots, F_r$.

We shall need the following lemma, which applies to a perfectly general field.

LEMMA. *Let Ψ be a closed irreducible system, in $u_1, \cdots, u_q$; $y_1, \cdots, y_p$, with $u_1, \cdots, u_q$ a set of arbitrary unknowns. Then there exists a basic set for Ψ*

$$A_1, \cdots, A_p$$

in which, if an A_i involves a u_j effectively, the partial derivative of A_i, with respect to the highest derivative of u_j in A_i, does not belong to Ψ.

We show first how to choose A_1. From among all forms of Ψ of class $q+1$, we select those of least rank in y_1. From the forms just selected, we choose such as have a least rank in u_q, and continue, taking the ranks in $u_{q-1}, \cdots, u_1$, in succession, as low as possible. For A_1, we take any of the forms thus obtained. Obviously A_1 fulfills our requirements. In choosing A_2 we first take all forms of Ψ of class $q+2$ which are reduced with respect to A_1. From these, we select such as have a least rank in y_2 and continue as above with respect to the u_i. We find thus an A_2 as specified. In the same way, we determine $A_3, \cdots, A_p$, in succession, to meet the requirements of the lemma.

36. Returning to the proof of our theorem, we limit ourselves, as, according to § 7, we may, to the consideration of the finite system (11). Introducing $r-1$ new unknowns, $v_2, \cdots, v_r$, we consider the system Σ_1, composed of the two forms

$$F_1, \; v_2 F_2 + \cdots + v_r F_r. \tag{12}$$

Let Ω_1 be used to represent the system (11) when the unknowns are the y_i, v_i.

Let Σ_1 be resolved into closed essential irreducible systems $A_1, \cdots, A_s$. Suppose that Ω_1 does not hold some A, say A_j. We say that, given any $n-1$ unknowns among $y_1, \cdots, y_n$, then A_j contains non-zero forms in those $n-1$ unknowns and the v_i. For instance, suppose that A_j does not contain a non-zero form in

$$y_1, \cdots, y_{n-1}; \qquad v_2, \cdots, v_r. \tag{13}$$

Then (13) will be a set of arbitrary unknowns for A_j, so that A_j will have a basic set consisting of one form, B, which introduces y_n. We take B algebraically irreducible. Then the general solution of B is the manifold of A_j.

We shall prove that B does not involve the v_i. For instance let B involve v_r.

According to § 22, the general solution of B is the same manifold for all arrangements of the unknowns. Thus far we have treated the unknowns as if y_n followed (13). Let us now give them the order

$$y_1, \cdots, y_n; \qquad v_2, \cdots, v_r.$$

Consider any regular solution of B. If we vary the y_i in this solution, and any finite number of their derivatives arbitrarily, but slightly, at some point ξ, we can, using the $v_2, \cdots, v_{r-1}$ of the given regular solution, determine v_r so as to get a second regular solution of B.* But this contradicts the fact that F_1 holds A_j. Thus B is free of the v_i.

This means that, given any solution $y_1, \cdots, y_n$ in the general solution of B considered as a form in the y_i alone, and given any analytic functions $v_2, \cdots, v_r$, the given y_i, v_i constitute a solution of A_j, hence a solution of

$$v_2 F_2 + \cdots + v_r F_r.$$

* Of course we have to construct new analytic functions y_i which assume, with their derivatives, the modified values at ξ.

But, as the v_i can be given arbitrarily, $F_2, \cdots, F_r$ must vanish separately for the given y_i. This means that Ω_1 holds $\mathcal{A}_j$. Our statement is proved.

Let $\mathcal{A}_1, \cdots, \mathcal{A}_q$ be those $\mathcal{A}_i$ which are not held by Ω_1. Consider any $n-1$ of the y_i,

$$(14) \qquad y_{i_1}, \cdots, y_{i_{n-1}}.$$

We extract from each $\mathcal{A}_i$, $i=1, \cdots, q$, a non-zero form in the y_i of (14) and the v_i. Let the q forms thus obtained be multiplied together. We obtain thus, for every set (14), a form which vanishes for every solution of Σ_1 which is not a solution of Ω_1. Now, as $\mathfrak{F}$ contains non-constant functions, we can so fix the v_i in $\mathfrak{F}$ that every form obtained above becomes a non-zero form in its set (14). Let the system of forms thus obtained, from the various sets (14), be denoted by Φ_1.

Let Π_1 represent the system of two forms in the y_i alone which Σ_1 becomes when the v_i are fixed definitely as above. Then every solution of Π_1 which is not a solution of Σ is a solution of Φ_1.

The unknowns v_i, whose rôle was episodic, now disappear from our discussion. We examine Π_1. We introduce $r-1$ new unknowns $w_2, \cdots, w_r$ and consider the system Σ_2 obtained by adjoining to Π_1, the form

$$w_2 F_2 + \cdots + w_r F_r.$$

Let Ω_2 be used to represent Σ, considered as a system in the y_i, w_i. Let Σ_2 be decomposed into closed essential irreducible systems $\mathcal{A}_1, \cdots, \mathcal{A}_s$. Suppose that Ω_2 does not hold some $\mathcal{A}$, say $\mathcal{A}_j$. We say that, given any $n-2$ of the y_i, then $\mathcal{A}_j$ contains a non-zero form in those $n-2$ y_i and the w_i. Imagine, for instance, that $\mathcal{A}_j$ does not contain a non-zero form in $y_1, \cdots, y_{n-2}$ and the w_i.

Every form of Φ_1 is in $\mathcal{A}_j$. For, let G be any form of Φ_1 and let F_k be any form of Ω_2 which does not hold $\mathcal{A}_j$. Consider any solution of $\mathcal{A}_j$. If the y_i in the solution annul F_k, they annul GF_k. If the y_i do not annul F_k, then, since

they annul each form of Π_1, they must, as seen above, annul G. Thus GF_k holds Λ_j, so that G is in Λ_j.

Thus Λ_j has a form in any $n-1$ of the unknowns y_i. Hence $y_1, \cdots, y_{n-2}$ and the w_i are a set of arbitrary unknowns for Λ_j, and Λ_j has a basic set B_1, B_2, which introduces y_{n-1} and y_n respectively. Let B_1 and B_2 be taken, as in the lemma of § 35, so that, if one of them involves a w_i, its derivative with respect to the highest derivative of that w_i is not in Λ_j. In addition, let B_1 be algebraically irreducible.

We say that B_1 and B_2 are free of the w_i. For instance, suppose that B_1 involves w_k effectively. Let G be the form of Φ_1 in $y_1, \cdots, y_{n-1}$. Then G vanishes for every solution in the general solution of B_1.* This, cannot be, for, ordering the unknowns in B_1 so that w_k comes last, we find that the $y_1, \cdots, y_{n-1}$, in any regular solution of B_1, can, together with any finite number of their derivatives, be given slight, but otherwise arbitrary, variations, at some point ξ, and w_k then be determined for a second regular solution of B_1.

Again, suppose that B_2 involves w_k. Let S be the derivative of B_2 with respect to the highest derivative of w_k in B_2.† Consider a regular solution of B_1, B_2 for which S does not vanish. Let H be the form of Φ_1 in $y_1, \cdots, y_{n-2}$, y_n alone. Let ξ be a point for which the functions in the solution and the coefficients of B_1, B_2, H are analytic, with the coefficients of H not all zero, and for which neither S nor the separants and initials of B_1, B_2 vanish. We can modify $y_1, \cdots, y_{n-2}$ slightly, but arbitrarily at ξ, and determine y_{n-1} so as to get a new regular solution of B_1. We can then use the modified $y_1, \cdots, y_{n-1}$ and, varying y_n and any finite number of its derivatives slightly, but arbitrarily, at ξ, determine w_k from $B_2 = 0$, securing another regular solution of B_1, B_2. Thus we can get a regular solution of B_1, B_2 which does not annul H.

* The remainder of G with respect to B_1 holds Λ_j and thus is 0.

† Notice that S is not the separant of B_2. We are using the unknowns in their original order.

Thus B_1 and B_2 are free of the w_i. Then the $y_1, \cdots, y_n$ in a regular solution of B_1, B_2, with arbitrary analytic functions $w_2, \cdots, w_r$, give a solution of Σ_2. This means that $F_2, \cdots, F_r$ all hold A_j, so that since F_1, as a form of Σ_2, holds A_j, Ω_2 holds A_j. This contradiction proves that A_j has a form in any $n-2$ of the y_i, and the w_i.

Let $A_1, \cdots, A_q$ be those systems A_i which are not held by Ω_2. Consider any $n-2$ of the y_i,

$$y_{i_1}, \cdots, y_{i_{n-2}}. \tag{15}$$

We extract from each A_i, $i = 1, \cdots, q$ a non-zero form in the y_i of (15), and multiply together the q forms thus obtained. We get, for every set (15), a form which is annulled by every solution of Σ_2 which is not a solution of Ω_2. We fix the w in $\mathfrak{F}$ so that each of the foregoing forms becomes a non-zero form in its unknowns (15).

Let Φ_2 be the set of forms thus obtained. Let Π_2 be the system which Σ_2 becomes when the w_i are fixed as above. Then every solution of Π_2 which is not a solution of Σ is a solution of Φ_2.

We form a system Σ_3, adjoining to Π_2 the form

$$z_2 F_2 + \cdots + z_r F_r$$

where the z_i are unknowns. We introduce Ω_3 in the expected way. Let A_j be a closed essential irreducible system held by Σ_3 which Ω_3 does not hold. We have to show that, given any $n-3$ of the y_i, there is a non-zero form in A_j in those y_i and the z_i alone. Suppose that A_j does not contain a non-zero form in $y_1, \cdots, y_{n-3}$ and the z_i. Then, as every form of Φ_2 holds A_j, A_j has a basic set B_1, B_2, B_3 which introduce y_{n-2}, y_{n-1}, y_n in succession. Let this basic set be selected as in the lemma of § 35. Furthermore, merely to abbreviate the proof, let us assume that B_1 is algebraically irreducible. We see at once that B_1 involves no z_i. Also, if B_2 involved a z_i, we could practice arbitrary slight variations on the $y_1, \cdots, y_{n-3}$, y_{n-1} and their derivatives

in a regular solution of B_1, B_2 and get a second such regular solution. This cannot be, since every such regular solution would have to be a solution of the form of Φ_2 in $y_1, \cdots, y_{n-3}$, y_{n-1}. Finally, if B_3 involved a z_i, we could take $y_1, \cdots, y_{n-3}$, y_n, and any finite number of derivatives, quite arbitrarily, at some point, and get a regular solution of B_1, B_2, B_3. This contradicts the fact that Φ_2 has a form in the above y_i alone.

Thus, the z_i being properly fixed, we get two systems, Π_3 and Φ_3, the latter containing a non-zero form in every $n-3$ of the y_i, such that Π_3 holds Σ and that every solution of Π_3 which is not a solution of Σ is a solution of Φ_3. In Π_3, there are four forms.

Continuing, we find a system equivalent to Σ, containing at most $n+1$ forms.

That $n+1$ equations may actually be necessary, in connection with n unknowns, is seen on considering the system in y,

$$y_1^2 - 4y, \qquad y_2 - 2,$$

which defines the general solution of $y_1^2 - 4y$. If a single form, G, had the manifold of this system, G would have to be of the first order in y. Then G would have to be divisible by $y_1^2 - 4y$, and so would admit the solution $y = 0$, which does not satisfy the given system.

37. The assumption above that $\mathfrak{F}$ does not consist entirely of constants is essential. For instance the system in y,

$$y_1, \quad y^2, \quad y-1$$

has no solutions. Still for any pair of constants d_1 and d_2, the form

$$d_1 y^2 + d_2(y-1)$$

has solutions in common with y_1.

However, the following result, which can doubtless be improved, holds for fields of constants.

THEOREM. *Let A be an algebraically irreducible form in a single unknown y, the order of A in y being r. Then*

there exists a system of forms, consisting of A and of at most r other forms, whose manifold is the general solution of A.

We shall need the following lemma:

LEMMA: *Let $\Sigma_1, \cdots, \Sigma_s$ be closed irreducible systems, none of which holds any other, and let Σ be a closed system which holds no Σ_i. Then there exists in Σ a form which holds no Σ_i.*

We proceed by induction. The lemma is true for $s = 1$. We shall prove that the truth for $s - 1$ implies the truth for s. The truth for $s - 1$ implies that each Σ_i, $i = 1, \cdots, s$, has a form A_i which holds no Σ_j with $j \neq i$. Let B_i, $i = 1, \cdots, s$, be a form of Σ which does not hold Σ_i. Let

$$P_i = A_1 \cdots A_{i-1} A_{i+1} \cdots A_s, \qquad i = 1, \cdots, s.$$

Consider the form

$$C = P_1 B_1 + \cdots + P_s B_s,$$

which belongs to Σ. Since $P_1 B_1$ does not hold Σ_1, and since P_i for $i > 1$ holds Σ_1, C does not hold Σ_1. Similarly, C holds no Σ_i.

Let A be resolved into closed essential irreducible systems,

$$\Sigma, \Sigma_1, \cdots, \Sigma_s,$$

Σ having the general solution of A for manifold. Let B_i, $i = 1, \cdots, s$ be a non-zero form of lowest rank in Σ_i, so that the manifold of Σ_i is the general solution of B_i. (We may and shall assume that each B_i is algebraically irreducible.) Each Σ_i is held by the separant S of A, hence by the resultant (as in algebra) with respect to y_r of A and S considered as polynomials in y_r. This means that each B_i is of order less than r in y.

Let A_1 be any form of Σ which holds no Σ_i. We shall examine the system A, A_1. This system is equivalent to the set of systems

$$\Sigma, \Sigma_1 + A_1, \cdots, \Sigma_s + A_1.$$

Let C_i be the remainder of A_1 with respect to B_i. Because A_1 does not hold Σ_i, $C_i \neq 0$. As C_i holds $\Sigma_i + A_1$, the resultant of B_i and C_i with respect to the highest derivative

in B_i holds $\Sigma_i + A_1$. Because B_i is algebraically irreducible and of higher rank than C_i, this resultant is not zero.

This means that if A, A_1 is resolved into closed essential irreducible systems,

$$\Sigma, \Sigma_1', \cdots, \Sigma_t',$$

each Σ_i' will contain a non-zero form of order less than $r-1$.*

Choosing now a form A_2 in Σ which holds no Σ_i', we form the system A, A_1, A_2 and operate as above. Continuing, we find that, after adjoining, to A, r or fewer forms of Σ, we get a system of forms whose manifold is that of Σ.

FORM QUOTIENTS

38. An expression A/B, where A and B are forms in $y_1, \cdots, y_n$, with B not identically zero, will be called a *form quotient*. Two form quotients will be considered equal if they are equal as rational functions of the y_{ij}. It is easy to see that, for A/B and C/D to be equal, it is necessary and sufficient that they yield the same analytic function for given analytic $y_1, \cdots, y_n$ which do not annul BD.

Let

$$(16) \qquad y = \frac{A}{B},$$

where A and B are forms in a single unknown u. The question which we shall study is that of attributing a meaning to y in the case in which u is such that both A and B vanish.

The totality Σ of forms in y und u which vanish for all solutions of

$$(17) \qquad By - A = 0$$

with $B \neq 0$ is an irreducible system. The manifold of Σ is the general solution of the equation obtained on dividing (17) by the highest common factor of A and B considered as polynomials in the u_i†.

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* If $r = 1$, this means that there are no Σ_i'.

† The results of Chapter VI will show that the manifold involved is independent of the field employed.

A function y will be said to *correspond to a function* u *through* (16) if u, y is a solution of Σ.

Example 1. Let

$$y = \frac{u_1}{u}.$$

Then every analytic y corresponds to u when $u = 0$. For, let α be any analytic function, not identically zero. If k is a non-zero constant, $y = \alpha_1/\alpha$ when $u = k\alpha$. Allowing k to approach zero, we find that $0, \alpha_1/\alpha$ belongs to the manifold of Σ.* By taking α suitably, we can make α_1/α become any desired analytic function (in some area).

Example 2. Let

$$y = \frac{u_1^2}{u}.$$

Referring to Example 2, § 12, we see that, since u_1 does not vanish for every solution in the general solution of $uy - u_1^2$, then $y^2 + u_1 y_1 - 2u_2 y$ must. Thus, for $u = 0$ in the general solution, we must have $y = 0$. If $u = k$, y approaches zero uniformly as k approaches zero. Thus $y = 0$, and no other function, corresponds to $u = 0$.

Example 3. Let

$$y = \frac{u}{u_1^2}.$$

We find that no y corresponds to $u = 0$.

Example 4. Let

$$y = \frac{(u_2^2 + u_1)(u_1 + u)}{u u_1}.$$

We find, putting $y - 1 = z$, that

(18) $$u u_1 z = u_2^2(u_1 + u) + u_1^2.$$

Differentiating, we have

(19) $$u_1^2 z + u u_1 z_1 = u_2 P,$$

where

(20) $$P = 2u_3(u_1 + u) + u_2(u_2 + u_1) + 2u_1 - uz.$$

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* See next to last paragraph of § 22.

Multiplying (19) by uz, and using (18), we find

$$u_1^3 z + u u_1^2 z_1 = u_2 Q \tag{21}$$

where

$$Q = u z P - u_2(u_1 z + u z_1)(u_1 + u). \tag{22}$$

We multiply (19) by u_1 and subtract from (21). Then

$$u_2(Q - u_1 P) = 0.$$

Then $Q - u_1 P$ holds Σ.

Suppose that u is a constant k distinct from 0. Then $Q - u_1 P = 0$ implies $Q = 0$. By (22), $zP = 0$, so that, by (20), $uz^2 = 0$. Thus $z = 0$ and $y = 1$. If we let $u = k + hx$, y approaches 1 as h approaches 0. Thus $y = 1$ and no other function, corresponds to u when u is a non-zero constant.

On the other hand, if we let $u = h\alpha$, we find that y approaches $(\alpha_1 + \alpha)/\alpha$ as h approaches 0. Thus every analytic function y corresponds to $u = 0$.

39. As the general solution of an equation is completely defined by a finite number of algebraic differential equations we see that, *if $\bar{u}$ makes A and B vanish, then either every analytic y corresponds to $\bar{u}$, or else, the functions y which correspond to $\bar{u}$ are the totality of solutions of a system of algebraic differential equations, in the coefficients of which, $\bar{u}$ figures.*

We are going to study the circumstances under which no y corresponds to a $\bar{u}$ which annuls A and B.

Writing $A = A(u)$, $B = B(u)$, we let

$$C(v) = A(\bar{u} + v), \quad D(v) = B(\bar{u} + v). \tag{23}$$

Then C and D are forms in v, with analytic coefficients which are not necessarily in $\mathfrak{F}$. Also, $v = 0$ makes C and D vanish. Let

$$C = E + H, \qquad D = F + K,$$

where E and F contain respectively those terms of C and D which are of lowest total degree in v and its derivatives.

Suppose first that the degree of E is at least that of F. Let α be an analytic function which does not annul F when substituted for v. Let v be replaced in C and D by $h\alpha$, where h is a constant. As h approaches 0, C/D will approach uniformly to an analytic function, which corresponds to $\bar{u}$ through (16).

If E is of lower degree than F, we see that $y = 0$ corresponds to $\bar{u}$ through $y = B/A$. In that case, we shall say that $y = \infty$ corresponds to $\bar{u}$ through (16). With this convention, every $\bar{u}$ for which A and B vanish has at least one corresponding y.

There appear to be grounds for conjecturing that if $\bar{u}$ annuls both A and B, and if more than one y corresponds to $\bar{u}$, then every analytic y corresponds to $\bar{u}$.

CHAPTER IV

SYSTEMS OF ALGEBRAIC EQUATIONS

40. The preceding chapters contain, of course, a theory of systems of algebraic equations in n unknowns, with analytic coefficients. One has only to suppose that the given system Σ consists of forms which are of order zero in each y_i. But there are good reasons why algebraic systems should receive special treatment.

To begin with, in most of the foregoing theory, algebraic equations are forced into an artificial association with differential equations. For instance, the closed essential irreducible systems held by a system of forms of order zero in each y_i, are systems of differential forms. One does not obtain thus Kronecker's theory of algebraic manifolds. We shall see that a purely algebraic theory of algebraic systems can be secured with the help of the notion of relative irreducibility which was studied in § 16.

But what is more important for us, from the standpoint of differential equation theory, is that the theory of algebraic systems can be developed from the algorithmic point of view, so that every entity whose existence is established is constructed with a finite number of operations. The results of the algebraic theory, when applied to systems of differential forms, will give us methods for determining the basic sets of the irreducible systems in a decomposition of a given finite system of differential forms. A theoretical process will be given for obtaining equations which completely define the irreducible systems. Also, we shall be able actually to construct the resolvents of irreducible systems of differential forms.

Finally, we shall apply the theory of algebraic systems to the study of the organic properties of the manifolds of systems of differential forms.

Our results relative to algebraic systems are mainly contained in the literature on algebraic manifolds.* For us, it will be convenient, in deriving these results, to use the methods of Chapters I and II.

Indecomposable systems of simple forms

41. We define a *domain of rationality* to be, as in algebra, a set of elements upon which the rational operations are performable, the set being closed with respect to such operations.† Every field is a domain of rationality.

Let a domain of rationality $\mathfrak{D}$ be given whose elements are functions of x, meromorphic in a given open region $\mathfrak{A}$.

By a *simple form*, we mean a form in $y_1, \cdots, y_n$ which is of order zero in each y_i.‡ Wherever the contrary is not stated, the coefficients in a simple form will be understood to belong to $\mathfrak{D}$.

A system Σ of simple forms will be said to be *simply closed* if every simple form which holds Σ belongs to Σ.

A system Σ of simple forms will be called *decomposable* if there exist two simple forms G and H such that neither G nor H holds Σ, while GH holds Σ. A system which is not decomposable will be called *indecomposable*.

Every system Σ of simple forms is equivalent to a finite set $\Sigma_1, \cdots, \Sigma_s$ of indecomposable systems. This is proved as in § 13,§ and, in fact, is an immediate application of the results

* Macaulay, *Modular Systems*. Van der Waerden, *Moderne Algebra*, vol. 2.

† The term "domain of rationality" is being displaced, in common usage, by the term "field". We have reserved the latter term for use as in the preceding chapters. See Dickson, *Algebras and their arithmetics*, Chapter XI.

‡ We prefer this term to *polynomial*, since we shall have to use the latter term in more general situations.

§ One can use here Hilbert's theorem on the existence of a finite basis for every system of polynomials in n variables, in place of the lemma of § 7.

on relative irreducibility in § 16.* The decomposition is unique in the sense of § 14.

SIMPLE RESOLVENTS

42. Let Σ be any non-trivial simply closed system. Then the unknowns can be divided into two sets, $u_1, \cdots, u_q$ and $y_1, \cdots, y_p$, $p+q=n$, such that no non-zero form of Σ is free of the y_i, while, for $j=1, \cdots, p$, there is a non-zero form of Σ in y_j and the u_i alone. We shall call the u_i a *set of unconditioned unknowns.* Let the unknowns be listed in the order

$$u_1, \cdots, u_q; \qquad y_1, \cdots, y_p,$$

and let

$$A_1, \cdots, A_p \tag{1}$$

be a basic set of Σ. Each A_i introduces y_i.

Then *every solution of* (1) *for which the initial of no* A_i *vanishes is a solution of* Σ. Furthermore, *if* Σ *is indecomposable, then* (1) *has regular solutions and every simple form which vanishes for the regular solutions of* (1) *is in* Σ.† These facts are evident.

43. Let Σ be a non-trivial simply closed system. We are going to show the existence of a simple form G, free of the y_i and of a form

$$Q = M_1 y_1 + \cdots + M_p y_p,$$

where the M_i are simple forms free of the y_i, such that, for two distinct solutions of Σ with the same u_i (if u_i exist), and with $G \neq 0$, Q gives two distinct functions of x.

By a *prime system,* we shall understand a simply closed indecomposable system.

Following § 25, we consider the system of forms obtained from Σ by replacing each y_i by a new unknown z_i. We take

* To satisfy all formalities, one can take, as $\mathfrak{F}$, the field obtained by first adjoining to $\mathfrak{D}$ the derivatives of all orders of the function sin $\mathfrak{D}$, and then forming all rational combinations of the functions in the enlarged set.

† The definition of regular solution is, of course, that of § 23.

the system Ω composed of the forms of Σ, the forms in the z_i just described, and also the form

$$\lambda_1(y_1-z_1)+\cdots+\lambda_p(y_p-z_p),$$

in which the λ_i are unknowns. Let $\varDelta$ be any prime system which Ω holds, and which does not contain every form y_i-z_i. We shall prove that $\varDelta$ contains a non-zero form which involves no unknowns other than the u_i and λ_i.

If $\varDelta$ contains a non-zero form in the u_i alone, we have our result. Suppose that $\varDelta$ contains no such form.

Since $\varDelta$ has all forms in Σ, $\varDelta$ has, for $j=1,\cdots,p$, a non-zero form B_j in y_j and the u_i alone. Then I_j the initial of B_j, since it involves only the u_i, is not in $\varDelta$. Similarly, let C_j, $j=1,\cdots,p$ be a non-zero form of $\varDelta$ in z_j and the u_i alone. Letting z_j follow the u_i in C_j, we see that the initial I_j' of C_j is not in $\varDelta$.

To fix our ideas, let us assume that y_1-z_1 is not in $\varDelta$. Consider any solution of $\varDelta$ for which

$$(y_1-z_1)I_1\cdots I_pI_1'\cdots I_p' \tag{2}$$

does not vanish. For such a solution, we have

$$\lambda_1=-\frac{\lambda_2(y_2-z_2)+\cdots+\lambda_p(y_p-z_p)}{y_1-z_1}.$$

Let m be the maximum of the degrees of the B_j in the y_j and of the degrees of the C_j in the z_j. Let α be any positive integer. We write, for $s=0,\cdots,\alpha$,

$$\lambda_1^s=\frac{F_s}{(y_1-z_1)^\alpha}$$

where F_s is a simple form. Now, it is plain that, using the relations $B_j=0$, $C_j=0$, we can depress the degree of F_s in each y_j and in each z_j to be less than m. The new expression for each λ_1^s will be of the form

$$\lambda_1^s=\frac{E_s}{(y_1-z_1)^\alpha D_s}$$

where D_s is a product of powers of the I_j and I_j'. Let D be the least common multiple of the D_s. We write

$$(3) \qquad \lambda_1^s = \frac{H_s}{(y_1 - z_1)^\alpha D},$$

$s = 0, \cdots, \alpha$, each H_s being a simple form of degree less than m in each y_j and z_j. Now the number of power products of the y_j, z_j of degree less than m in each y_j and z_j is m^{2p}. Consequently, if we take $\alpha \geqq m^{2p}$, we can find a non-zero polynomial in λ_1, of degree not greater than α, whose coefficients are simple forms in $\lambda_2, \cdots, \lambda_p$ and the u_i, which vanishes for every solution of Λ for which (2) does not vanish. The form in the λ_i, u_i thus obtained belongs to Λ.

The existence of G and Q is then proved as in § 25. We notice that, since we are dealing with simple forms, it is possible to take the M_i here, which correspond to the μ_i of § 25, and to the M_i of § 26, as integers; in short, no derivatives of the λ_i will appear in the forms K, L of § 25.

44. Let Σ be any non-trivial prime system.

We take a pair G, Q as in § 43.

We introduce a new unknown w, and form a system Λ by adjoining $w - Q$ to Σ. Let Ω be the system of all simple forms in w, the u_i and y_i which vanish for all solutions of Λ. It is easy to prove, as in § 28, that Ω is indecomposable. Those forms of Ω which are free of w are precisely the forms of Σ.

As above, we prove that Ω has a non-zero form free of the y_i.

We now arrange the unknowns in Ω in the order

$$u_1, \cdots, u_q; \quad w; \quad y_1, \cdots, y_p$$

and take a basic set for Ω

$$(4) \qquad A, A_1, \cdots, A_p.$$

Here, $w, y_1, \cdots, y_p$ are introduced in succession.

We take A algebraically irreducible relative to $\mathfrak{D}$.

As in § 29, it follows that each A_i is linear in y_i so that the equation $A_i = 0$ expresses y_i rationally in w and the u_j. We call the equation $A = 0$ a *simple resolvent* of Σ (or of any system of simple forms equivalent to Σ).*

It is easy now to prove that q, in § 42, is independent of the manner in which the u_i are selected.

BASIC SETS OF PRIME SYSTEMS

45. We consider simple forms in the unknowns

$$u_1, \cdots, u_q; \quad y_1, \cdots, y_p.$$

Let

$$(5) \qquad A_1, A_2, \cdots, A_p$$

be an ascending set of simple forms, each A_i being of class $q+i$. We are going to find a condition for (5) to be a basic set for a prime system.

In what follows immediately, we consider the u_i to be complex variables, and the y_i to be functions of the u_i and x. We represent A_i, with this interpretation of the symbols in it, by a_i.

We denote by $\mathfrak{B}$ an open region in the space of x; $u_1, \cdots, u_q$, for every point of which x lies in $\mathfrak{A}$.

We are going to prove that, *for* (5) *to be a basic set of a prime system, it is necessary and sufficient that*

(a) *Given any open region $\mathfrak{B}$, there exist p functions, $\zeta', \zeta'', \cdots, \zeta^{(p)}$ of x; $u_1, \cdots, u_q$, analytic in some open region contained in $\mathfrak{B}$, which make each a_i vanish when they are substituted for $y_1, \cdots, y_p$ respectively, and*

(b) *for every $i \leqq p$, given any analytic functions $\zeta', \cdots, \zeta^{(i-1)}$ of x; $u_1, \cdots, u_q$ which cause $a_1, \cdots, a_{i-1}$ to vanish when substituted for $y_1, \cdots, y_{i-1}$, the coefficient of the highest power of y_i in a_i does not vanish for $y_j = \zeta^{(j)}, j = 1, \cdots \cdots, i-1$, and, after these substitutions, a_i, as a polynomial in y_i, is irreducible in the domain of rationality obtained*

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* One is equipped now to read the first part of Chapter VII.

*by adjoining to $\mathfrak{D}$ the variables $u_1, \cdots, u_q$ and the functions $\zeta', \cdots, \zeta^{(i-1)}$.**

Furthermore, we shall see that if (a) and (b) are fulfilled, no non-zero polynomial in the u_i, y_i, with coefficients in $\mathfrak{D}$, of lower degree than each a_j in y_j, $j = 1, \cdots, p$ can vanish for $y_j = \zeta^{(j)}$, $j = 1, \cdots, p$ where the $\zeta^{(j)}$ are functions as in (a).

46. In §§ 46, 47, we treat the necessity of the conditions. We assume the existence of a prime system Σ for which (5) is a basic set.

Let m_i be the degree of a_i in y_i.

Let $\mathfrak{D}_0$ be the domain of rationality obtained by adjoining the u_j to $\mathfrak{D}$. If a_1 were reducible in $\mathfrak{D}_0$, A_1 would be the product of two forms, U and V, each of degree less than m_1 in y_1. As one of U, V would hold Σ, (5) could not be a basic set.

Then a_1 is irreducible, so that the equation $a_1 = 0$ determines y_1, in some open region $\mathfrak{B}_1$, contained in $\mathfrak{B}$, as any one of m_1 distinct analytic functions $\zeta_1', \cdots, \zeta_{m_1}'$ of the variables $x; u_1, \cdots, u_q$.

As the coefficient of the highest power of y_2 in a_2 is of lower degree in y_1 than a_1, that coefficient cannot vanish identically in $x; u_1, \cdots, u_q$ if y_1 is replaced by any ζ_i'.

Let y_1 be replaced by some ζ_i' in a_2 and let α_2 be the polynomial in y_2 which is thus obtained from a_2. Let $\mathfrak{D}_i$ be the domain of rationality obtained on adjoining the indicated ζ_i' to $\mathfrak{D}_0$.

Suppose that α_2 as a polynomial in y_2, is reducible in $\mathfrak{D}_i$. Let $\alpha_2 = \varphi_1 \varphi_2$ with φ_1 and φ_2 polynomials in y_2, of positive degree, with coefficients in $\mathfrak{D}_i$. Each coefficient in φ_1 and φ_2 is of the form δ/β, where δ and β are polynomials in $u_1, \cdots, u_q; \zeta_i'$ with coefficients in $\mathfrak{D}$.

Let θ be the product of all denominators β. We may write

$$\theta \alpha_2 = \psi_1 \psi_2 \tag{6}$$

* Thus, the enlarged domain of rationality is a set of functions of $x; u_1, \cdots, u_q$. For $i = 1$, the above is to mean that a_1, as a polynomial in y_1, is irreducible when $u_1, \cdots, u_q$ are adjoined to $\mathfrak{D}$.

The above result establishes an equivalence between the basic sets of prime systems and certain sets of polynomials used by van der Waerden in

where ψ_1 and ψ_2 are polynomials in y_2 of degree less than m_2, whose coefficients are polynomials in $u_1, \cdots, u_q; \zeta_i'$. Making use of the relation $a_1 = 0$ for ζ_i' we depress the degrees in ζ_i' of the coefficients in ψ_1, ψ_2 to less than m_1. Each coefficient will be of the form η / γ, with γ a polynomial in $u_1, \cdots, u_q$. Multiplying through in (6) by θ_1, the product of the denominators γ, one obtains a relation

$$(7) \qquad \theta_1 \theta \alpha_2 = \xi_1 \xi_2$$

with ξ_1 and ξ_2 polynomials in y_2 of degree less than m_2. The coefficients of ξ_1, ξ_2 are polynomials in $\zeta_i'; u_1, \cdots, u_q$, of degree less than m_1 in ζ_i'. We notice that neither ξ_1 nor ξ_2 vanishes identically in $x; u_1, \cdots, u_q; y_2$. Let t, g_1, g_2 be the polynomials which result respectively from $\theta_1 \theta, \xi_1, \xi_2$ on replacing ζ_i' by y_1. Then

$$(8) \qquad t a_2 - g_1 g_2$$

vanishes for $y_1 = \zeta_i'$. Let s_1 be the coefficient of $y_1^{m_1}$ in a_1. We obtain a relation

$$(9) \qquad s_1^{\mu} (t a_2 - g_1 g_2) - k a_1 = b$$

with k and b polynomials in $u_1, \cdots, u_q, y_1, y_2$, and b of degree less than m_1 in y_1.

Now b vanishes identically in $u_1, \cdots, u_q; y_2$ for $y_1 = \zeta_i'$. Hence, if b is written as a polynomial in y_2 each coefficient must vanish for $y_1 = \zeta_i'$. As a_1 is irreducible in $\mathfrak{D}_0$, and as the coefficients in b are of degree less than m_1 in y_1, b must be identically zero.

Let I_1 be the initial of A_1, and let T, G_1, G_2, K be the forms which t, g_1, g_2, k become when the u_i, y_i are regarded as unknowns. Then

$$(10) \qquad I_1^{\mu} (T A_2 - G_1 G_2) - K A_1 = 0.$$

We observe that G_1 and G_2 are reduced with respect to A_1, A_2 and are not zero. Now (10) shows that $I_1^{\mu} G_1 G_2$

his treatment of prime ideals of polynomials. (Mathematische Annalen, vol. 96, 1927, p. 189.)

holds Σ. Thus one of G_1, G_2 must hold Σ, which is impossible.

Hence α_2 is irreducible in $\mathfrak{D}_i$.

47. Thus the equation $\alpha_2 = 0$ defines y_2 as one of m_2 functions ζ_j'', $j = 1, \cdots, m_2$, of x; $u_1, \cdots, u_q$, each analytic in some open region $\mathfrak{B}_2$, contained in $\mathfrak{B}_1$. Evidently we can use a single open region $\mathfrak{B}_2$ which will serve no matter which ζ_i' is used in determining the m_2 functions ζ_j''. That is, we have $m_1 m_2$ pairs ζ_i', ζ_j'' which are solutions for y_1, y_2 of $a_1 = 0$, $a_2 = 0$ and these $m_1 m_2$ pairs are the only solutions of $a_1 = 0$, $a_2 = 0$ analytic in $\mathfrak{B}_2$.*

Consider any pair ζ_i', ζ_j''. Let s_3 be the coefficient of the highest power of y_3 in a_3. We shall show that s_3 does not vanish when $y_1 = \zeta_i'$, $y_2 = \zeta_j''$. Suppose that s_3 does vanish. Let σ be the polynomial in y_2 which s_3 becomes when $y_1 = \zeta_i'$. As α_2 is irreducible in $\mathfrak{D}_i$, we see, since s_3 is of degree less than m_2 in y_2, that the coefficients of the powers of y_2 in σ are zero. That is, the coefficients of the powers of y_2 in s_3 vanish for $y_1 = \zeta_i'$. But those coefficients are of lower degree than m_1 in y_1. This proves our statement.

Let $\mathfrak{D}_{ij}$ be obtained from $\mathfrak{D}_0$ by the adjunction of ζ_i' and ζ_j''. Let α_3 be the polynomial in y_3 which a_3 becomes for $y_1 = \zeta_i'$, $y_2 = \zeta_j''$. We shall prove that α_3, as a polynomial in y_3, is irreducible in $\mathfrak{D}_{ij}$.

Suppose that it is not. Then $\alpha_3 = \varphi_1 \varphi_2$ with φ_1 and φ_2 polynomials in y_3 of degree less than m_3, with coefficients in $\mathfrak{D}_{ij}$. Each coefficient in φ_1 and φ_2 is of the form δ/β, where δ and β are polynomials in $u_1, \cdots, u_q$; ζ_i', ζ_j'', with coefficients in $\mathfrak{D}$.

Let θ be the product of the denominators β. Then

$$\theta \alpha_3 = \psi_1 \psi_2 \tag{11}$$

where ψ_1 and ψ_2 are polynomials in y_3 of degree less than m_3, whose coefficients are polynomials in $u_1, \cdots, u_q$; ζ_i', ζ_j''.

Making use of the relation $\alpha_2 = 0$ for ζ_j'', we depress each coefficient in ψ_1, ψ_2 to be of degree less than m_2

* The sets of m_2 functions ζ_j'' corresponding to two distinct ζ_i' may have functions in common.

in ζ_j''. The new coefficients will be of the form γ/η with η a power of the coefficient of $y_2^{m_2}$ in α_2 and γ a polynomial in $u_1, \cdots, u_q$; ζ_i', ζ_j'' of lower degree than m_2 in ζ_j''. Thus if θ_1 is the highest of the powers η, we have

$$\theta_1 \, \theta \, \alpha_3 = \xi_1 \, \xi_2$$

with ξ_1, ξ_2 polynomials whose degrees in y_3, ζ_j'' are respectively lower than m_3, m_2. In the same way we depress the degrees in ζ_i' of the coefficients in ξ_1 and ξ_2 to be less than m_1. We find thus

$$\theta_2 \, \theta_1 \, \theta \, \alpha_3 = \tau_1 \, \tau_2$$

with θ_2 a power of the coefficient of $y_1^{m_1}$ in a_1, and with τ_1, τ_2 polynomials in $u_1, \cdots, u_q$; ζ_i', ζ_j''; y_3 whose degrees in y_3, ζ_j'', ζ_i' are respectively less than m_3, m_2, m_1. Furthermore, neither of τ_1, τ_2 vanishes identically in

$$x; \; u_1, \; \cdots, \; u_q; \; y_3.$$

Let t, g_1, g_2 result respectively from $\theta_2 \, \theta_1 \, \theta$, τ_1 and τ_2 on replacing ζ_i' by y_1 and ζ_j'' by y_2. Then

$$t \, a_3 - g_1 \, g_2$$

vanishes for $y_1 = \zeta_i'$, $y_2 = \zeta_j''$. Let s_1 and s_2 be respectively the coefficients of $y_1^{m_1}$ in a_1 and of $y_2^{m_2}$ in a_2. Then we have a relation

$$(12) \qquad s_1^{\mu_1} s_2^{\mu_2} (t \, a_3 - g_1 \, g_2) - k_1 \, a_1 - k_2 \, a_2 = b$$

with k_1, k_2, b polynomials in $u_1, \cdots, u_q$; y_1, y_2, y_3, the degrees of b in y_1 and y_2 being less than m_1 and m_2 respectively.

Now b vanishes identically in $u_1, \cdots, u_q$; y_3 if y_1 and y_2 are replaced by ζ_i' and ζ_j'' respectively. Hence, if b is written as a polynomial in y_3, each coefficient must vanish for $y_1 = \zeta_i'$, $y_2 = \zeta_j''$. Considering the degrees of the coefficients in y_1 and y_2, we see by the argument used in proving that s_3 does not vanish, that b vanishes identically. Thus (12) gives a relation

$$(13)\qquad I_1^{\mu_1} I_2^{\mu_2}(TA_3 - G_1 G_2) - K_1 A_1 - K_2 A_2 = 0$$

with I_i the initial of A_i, and with G_1, G_2 not zero and reduced with respect to A_1, A_2, A_3.

This proves that α_3 is irreducible in $\mathfrak{D}_{ij}$.

Continuing, we prove the necessity of the conditions stated in § 45.*

48. We turn now to the sufficiency proof. Let the conditions stated in § 45 be satisfied. We shall prove that (5) has solutions for which no initial vanishes and that, if G and H are simple forms such that GH vanishes for all solutions of (5) which make no initial zero, then either G vanishes for all such solutions, or else H does.

Let ζ', $\cdots$, $\zeta^{(p)}$ be functions as in § 45. Then no I_j vanishes when the y_i are replaced by the $\zeta^{(i)}$. Let

$$(14)\qquad x_0;\ \xi_1,\ \cdots,\ \xi_q$$

be values of x; u_1, $\cdots$, u_q for which the $\zeta^{(i)}$ are analytic, the coefficients in the A_i being analytic at x_0 and no I_i vanishing for the above values. If we take $u_i = \xi_i$, $i = 1, \cdots, q$, the $\zeta^{(i)}$ become functions of x which constitute a solution of (5) with $I_1 \cdots I_p \neq 0$.

Let now G and H be such that GH vanishes for all solutions with $I_1 \cdots I_p \neq 0$. Let G_1 and H_1 be, respectively, remainders for G and H with respect to (5). Then $G_1 H_1$ vanishes for all solutions with $I_1 \cdots I_p \neq 0$. Then $G_1 H_1$ must vanish identically in x; u_1, $\cdots$, u_q when the y_i are replaced by the $\zeta^{(i)}$ as above. This is because, if the quantities (14) are varied slightly, but otherwise arbitrarily, the $\zeta^{(i)}$ will still give a solution of (5) with $I_1 \cdots I_p \neq 0$. Hence either G_1 or H_1 vanishes for the above replacements. Suppose that G_1 does. Then G_1, being reduced with respect to (5), vanishes identically. Thus G vanishes for all solutions for which no initial vanishes, and we have our result.

* One might replace the above proof by an induction proof, in which irreducibility is proved for only one α_i with $i > 1$. We think that, on the whole, the above treatment is less oppressive than one by induction.

Let Σ be the totality of simple forms which vanish for the solutions of (5) for which no initial vanishes. Then Σ is simply closed, and indecomposable. Now, if Σ contained a non-zero form G, reduced with respect to (5), G would vanish for $y_i = \zeta^{(i)}$, $i = 1, \cdots, p$. This is impossible. Then (5) is a basic set for Σ.

Of course, Σ contains no non-zero form in the u_i alone. Also, by the methods of elimination frequently used, it can be shown that, for $j = 1, \cdots, p$, Σ has a non-zero form in y_j and the u_i alone.

49. Let (5) be a basic set of a prime system Σ. We have seen that every solution of (5) for which no initial vanishes, is a solution of Σ. We shall now prove that *every solution of* (5) *for which no separant vanishes is a solution of* Σ.

Consider any solution of (5)

$$\bar{u}_1, \cdots, \bar{u}_q; \bar{y}_1, \cdots, \bar{y}_p \tag{15}$$

for which no S_i vanishes. Let G be any form in Σ. Let x_0 be a point at which the functions in (15) and the coefficients in the A_i and G are analytic, and for which no S_i vanishes for (15). Let

$$\xi_1, \cdots, \xi_q; \eta_1, \cdots, \eta_p$$

be the values of (15) at x_0. An easy application of the implicit function theorem shows that we can get functions $\zeta', \cdots, \zeta^{(p)}$ as in § 45, analytic at

$$x_0; \xi_1, \cdots, \xi_q \tag{16}$$

and assuming there the values $\eta_1, \cdots, \eta_p$ respectively.

If we put $u_i = \bar{u}_i$ in $\zeta^{(j)}$, $i = 1, \cdots, q$, $\zeta^{(j)}$ becomes $\bar{y}_j$. Since the $\zeta^{(j)}$ make no I_i zero, we can find values

$$x_0'; \xi_1', \cdots, \xi_q' \tag{17}$$

as close as we please to (16), which, with the corresponding values $\eta_1', \cdots, \eta_p'$ of the $\zeta^{(j)}$, make no I_i zero. If we take $u_i = \xi_i'$, $i = 1, \cdots, q$, the $\zeta^{(j)}$ give p analytic functions y_j which, with the u_i, constitute a solution of (5) for which no initial vanishes. Hence G vanishes for

$$x_0'; \quad \xi_1', \cdots, \xi_q'; \quad \eta_1', \cdots, \eta_p',$$

if x_0' is sufficiently close to x. By continuity, G vanishes for

$$x_0; \quad \xi_1, \cdots, \xi_q; \quad \eta_1, \cdots, \eta_p.$$

This means that G vanishes for (15). Our result is proved.

Construction of resolvents

50. Before we can develop a method for the effective construction of a resolvent for a prime system for which a basic set is given, we must have a solution of the following problem.

Let A be a simple form in $u_1, \cdots, u_q$; w, of positive degree in w, irreducible as a polynomial in w in $\mathfrak{D}_0$ (§ 46). Let A_1 be a simple form in $u_1, \cdots, u_q$; w; y, of positive degree in y. Let ζ_1 be any analytic function of x; $u_1, \cdots, u_q$ which renders A zero when substituted for w. Let α be the polynomial in y obtained by replacing w by ζ_1 in A_1. We assume that the initial of A_1 does not vanish for $w = \zeta_1$. It is required to determine the irreducible factors of α in $\mathfrak{D}_1$, the domain of rationality obtained by adjoining ζ_1 to $\mathfrak{D}_0$.

Several methods are known for resolving α into its irreducible factors. The following treatment is taken from van der Waerden's *Moderne Algebra*, vol. 1, p. 130, where a more general algebraic situation is considered.

It must not be thought that we must actually possess ζ_1 to carry out the factorization. It will be seen that all operations used are rational, and that we get expressions for the factors of α with no knowledge relative to ζ_1 except that it renders A zero.

Let A be of degree m in w and let $\zeta_2, \cdots, \zeta_m$ be the analytic functions of x; $u_1, \cdots, u_q$, other than ζ_1, which render A zero when substituted for w. We assume all ζ_i to be analytic in some open region $\mathfrak{B}$.

Let z be an indeterminate and let β_1 be the polynomial in y and z which results on replacing y in α by $y - z\zeta_1$. Let β_i, $i = 2, \cdots, m$ result from β_1 on replacing ζ_1 by ζ_i, $i = 2, \cdots, m$. Let $\gamma = \beta_1 \beta_2 \cdots \beta_m$.

Then γ is a polynomial in y, z with coefficients in $\mathfrak{D}_0$, the coefficients being capable of determination by the theory of symmetric functions. Let γ be resolved into irreducible factors in $\mathfrak{D}_0$. This is possible, provided that we are able to factor a polynomial in one variable with coefficients in $\mathfrak{D}$.* Let

$$\gamma = \delta_1 \cdots \delta_r \tag{18}$$

with each δ_i a polynomial of positive degree in y, z, with coefficients in $\mathfrak{D}_0$ and irreducible in $\mathfrak{D}_0$. Finally let τ_i, $i = 1, \cdots, r$ be the highest common factor of β_1 and δ_i, both considered as polynomials in y, z, the domain of rationality being $\mathfrak{D}_1$. This highest common factor is obtained by the Euclid algorithm, bearing in mind that a polynomial ξ in $\zeta_1, u_1, \cdots, u_q$ is zero when and only when the polynomial c, in w, obtained by replacing ζ_1 by w, in ξ, is the product by A of a polynomial in w with coefficients in $\mathfrak{D}_0$.

We shall prove that the highest common factors just found become, for $z = 0$, the irreducible factors of α in $\mathfrak{D}_1$. Let

$$\alpha = \varphi_1 \varphi_2 \cdots \varphi_k$$

be a resolution of α into irreducible factors. Then

$$\beta_1 = \psi_1 \psi_2 \cdots \psi_k$$

where each ψ_i results from φ_i on replacing y by $y - z\zeta_1$. It is easy to see that each ψ_i, as a polynomial in y, z, is irreducible in $\mathfrak{D}_1$.

Manifestly each ψ_i is a common factor of β_1 and of some δ_j in (18). If we can prove that, in this case, ψ_i is the *highest common factor* of β_1 and δ_j, we will have our result.

Let $\psi_i^{(j)}$, for $j = 2, \cdots, m$, be the polynomial obtained from ψ_i on replacing ζ_1 by ζ_j. Let

$$\eta_i = \psi_i \psi_i'' \cdots \psi_i^{(m)}. \tag{19}$$

Then η_i is a polynomial in y, z with coefficients in $\mathfrak{D}_0$, and

$$\gamma = \eta_1 \eta_2 \cdots \eta_k.$$

Each δ_i in (18) is a factor of some η_j.

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* Perron, *Algebra*, vol. 1, p. 210.

Suppose, that ψ_1 is a factor of δ_1 and that δ_1 is a factor of η_1. If we prove that ψ_1 is the highest common factor of β_1 and η_1 we shall have our result.

Suppose, for instance, that η_1 is divisible by $\psi_1 \psi_2$. Then, by (19)

$$\psi_1'' \cdots \psi_1^{(m)} = \varrho(y, z)\, \psi_2, \tag{20}$$

where ϱ is a polynomial in y, z, with coefficients in $\mathfrak{D}_1$.

The set of terms of highest degree in y, z in the first member of (20) is of the form

$$b(y - z\zeta_2)^s \cdots (y - z\zeta_m)^s, \tag{21}$$

with b a rational combination of the u_i and ζ_i. The terms of highest degree in the second member give an expression of the type

$$\sigma(y, z)\,(y - z\zeta_1)^t. \tag{22}$$

Now (21) and (22) cannot be equal, since no $y - z\zeta_i$ with $i > 1$ is divisible by $y - z\zeta_1$. This completes the proof.

51. We consider a non-trivial prime system Σ in the unknowns u_i, y_i, for which

$$A_1, A_2, \cdots, A_p \tag{23}$$

is a basic set, each A_i introducing y_i. In §§ 53, 54 we show how, when the A_i are given, a simple resolvent can be constructed for Σ.

52. Let $\lambda_1, \cdots, \lambda_p$ be new unknowns. Let Σ_1 be used to represent Σ when the unknowns are the u_i, λ_i, y_i. It is easy to show, as in § 17, that Σ_1 is indecomposable. Furthermore, no non-zero form in the u_i, λ_i holds Σ_1.

We see as in § 43 (or § 25), that there exists a non-zero form G in the u_i, λ_i such that, for two distinct solutions of Σ_1 with the same u_i, λ_i, for which G does not vanish, the form

$$Q = \lambda_1 y_1 + \cdots + \lambda_p y_p$$

gives two distinct functions of x.*

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* At present, we have no way of determining G.

By § 44, a simple resolvent exists for Σ_1, for which $w = Q$. Let Ω be the system of all simple forms in the u_i, λ_i, w, y_i which hold Σ_1 and $w - Q$. We consider a basic set for Ω,

$$R, R_1, \cdots, R_p \tag{24}$$

in which $w, y_1, \cdots, y_p$ are introduced in succession and in which R is algebraically irreducible. Then $R = 0$ is a resolvent for Σ_1 and each R_i is linear in y_i.

53. We shall now show how a basic set (24) can actually be constructed.

By the method of elimination of § 31, we can determine, by a finite number of rational operations, a non-zero simple form S in w, the λ_i and u_i which vanishes for all solutions of (23) and $w - Q$ for which no initial in (23) vanishes. Then S belongs to Ω. Now, let

$$S = S_1 \cdots S_r$$

with each S_i algebraically irreducible relative to $\mathfrak{D}$. Then some S_i holds Ω. The selection of such an S_i can be made as follows. Consider any S_i and let T be the form obtained from it on replacing w by Q. For S_i to hold Ω, it is necessary and sufficient that T hold Σ_1. Let T be arranged as a polynomial in the λ_i. For T to hold Σ_1, it is necessary and sufficient that each coefficient in the polynomial hold Σ. A coefficient will hold Σ if and only if its remainder with respect to (23) is zero.

Every form in the u_i, λ_i and w which holds Ω is divisible by R. Thus an irreducible factor of S which holds Ω must be the product of R in (24) by a function in $\mathfrak{D}$.

We have then a method for constructing a simple resolvent for Σ_1. It remains to show how a complete basic set (24) can be determined.

Let U be the form which results from R on replacing w by $w + y_1$ and λ_1 by $\lambda_1 + 1$. Then U holds Ω. The degree of U in y_1 is that of R in w and the coefficient of the highest power of y_1 in U is free of w.

Now let ζ represent any analytic function of the u_i, λ_i and x which makes R vanish when substituted for w. Let α be the polynomial in y_1 obtained on replacing w in U by ζ. Let

$$(25) \qquad \alpha = \alpha_1 \alpha_2 \cdots \alpha_m$$

be a decomposition of α into irreducible factors obtained as in § 50. The coefficients in the α_i are rational combinations of ζ, the u_i and λ_i. Let β be the product of the denominators of these coefficients. Then

$$\beta\alpha = \gamma_1 \gamma_2 \cdots \gamma_m.$$

The γ_i are irreducible and their coefficients are polynomials in ζ, the u_i and λ_i. Let B be the form which results from β on replacing ζ by w. Let C_i result similarly from γ_i. Then

$$(26) \qquad BU - C_1 \cdots C_m$$

vanishes identically in y_1 when w is replaced by ζ. It follows, as in § 46, that (26) holds Ω, hence that some C_i holds Ω.

Suppose that C_1 is found (by test) to hold Ω. We say that C_1 is linear in y_1. If I_1 is the initial of R_1 in (24) we have

$$(27) \qquad I_1^{\mu} C_1 = HR_1 + K$$

where K is free of y_1. As K holds Ω, it is divisible by R. Thus, if C_1 were not linear, (27) would imply that γ_1 is reducible.*

It is only necessary, then, to take the remainder of C_1 with respect to R in order to have a form which will serve as R_1 in (24).

The R_i with $i > 1$ in (24) are determined in the same way.

54. It remains now to construct a resolvent for Σ.

Let I be the initial of R, in (24) and I_i that of R_i. As I_i and R are relatively prime polynomials, we can find forms M_i, N_i, L_i such that

* We note that I_1 cannot vanish when w is replaced by ζ.

(28) $$M_i I_i + N_i R = L_i$$
with L_i free of w, and not zero.

Let the λ_i be replaced by integers a_i in such way that

$$I I_1 \cdots I_p L_1 \cdots L_p$$

does not vanish. We shall show how (24) gives, for these substitutions, a resolvent for Σ with

$$w = a_1 y_1 + \cdots + a_p y_p.$$

Let Φ be the indecomposable system obtained by adjoining

$$w - a_1 y_1 - \cdots - a_p y_p$$

to Σ. For the substitution $\lambda_i = a_i$, (24) becomes a system

(29) $$R',\ R'_1,\ \cdots,\ R'_p.$$

Then each form of (29) holds Φ. Let R' be resolved into its irreducible factors in $\mathfrak{F}$. One factor S, which can be determined, will hold Φ.

If we put $\lambda_i = a_i$ in (28), we see that no R'_i has an initial which is divisible by S. Let S_i be the remainder of R'_i with respect to S. Then each form in the set

(30) $$S,\ S_1,\ \cdots,\ S_p$$
holds Φ.

It is clear that (30) is a basic set for the totality of simple forms which hold Φ. To show that $S = 0$ is a simple resolvent for Σ, we have to prove the existence of the form G of § 43. If two distinct solutions of Φ have the same u_i and w, the u_i and w must make the initial I_i' of some S_i in (30) vanish. Aus I'_i and S are relatively prime polynomials, we have a relation

$$M_i I'_i + N_i S = L_i$$

with L_i a non-zero form in the u_i alone. Then $L_1 \cdots L_p$, which can actually be constructed, will serve as G.

Resolution of a finite system into indecomposable systems

55. Let Σ be any finite system of simple forms in $y_1, \cdots, y_n$, not all zero. In this section, we show how to determine basic sets of a finite number of prime systems which form a set of systems equivalent to Σ. Later, we shall obtain finite systems of forms equivalent to the prime systems.

Let

$$(31) \qquad A_1, A_2, \cdots, A_p$$

be a basic set of Σ, determined as in § 4. If A_1 is of class zero, Σ has no solutions and is thus indecomposable. We assume now that A_1 is not of class zero. For every form in Σ, let the remainder with respect to (31) be determined. If these remainders are adjoined to Σ, we get a system Σ' equivalent to Σ. By § 4, if not all remainders are zero, Σ' will have a basic set of lower rank than (31). We see, by § 3, that after a finite number of repetitions of the above operation, we arrive at a finite system Λ, equivalent to Σ, with a basic set (31) for which either A_1 is of class zero or for which otherwise the remainder of every form in Λ is zero.

Let us suppose that we are in the latter case. We shall make a temporary relettering of the y_i. If, in the basic set (31) for Λ, A_i is of class j_i, we replace the symbol y_{j_i} by y_i. The $q = n - p$ unknowns not among the y_{j_i} we call, in any order, $u_1, \cdots, u_q$. We list all the unknowns in the order $u_1, \cdots, u_q;\ y_1, \cdots, y_p$.

With this change of notation, we proceed to determine, using § 45, whether (31) is a basic set for a prime system.

If A_1 is reducible, as a polynomial in y_1 and if $A_1 = B_1 B_2$, where B_1 and B_2 are of positive degree in y_1, then Λ will be equivalent to $\Lambda + B_1$, $\Lambda + B_2$ and each of the latter systems, after we revert to the old notation for the unknowns, will have a basic set lower than (31).

Suppose then that A_1 is irreducible and let ζ' be any analytic function of $x;\ u_1, \cdots, u_q$ which annuls A_1 when substituted for y_1. Let α_2 be the polynomial in y_2 which

A_2 becomes, for this substitution. Suppose that α_2 is reducible as a polynomial in y_2. By (10), there exist non-zero forms G_1, G_2, reduced with respect to A_1, A_2, such that $I_1 G_1 G_2$ holds $\mathit{\Lambda}$. Of course, § 50 furnishes a method for actually determining G_1 and G_2. Then $\mathit{\Lambda}$ is equivalent to

$$\mathit{\Lambda} + I_1, \quad \mathit{\Lambda} + G_1, \quad \mathit{\Lambda} + G_2.$$

Each of the latter systems, after we revert to the old notation for the unknowns, has a basic set of lower rank than (31). This becomes clear if one considers that, in (10), T, G_1, G_2, K do not involve any u_i not effectively present in A_1 and A_2.

Suppose now that α_2 is irreducible. Let A_1 and A_2 vanish for $y_1 = \zeta'$, $y_2 = \zeta''$. Let α_3 result in the usual manner from A_3. If α_3 is reducible with respect to y_3, we see from (13) that $\mathit{\Lambda}$ is equivalent to

$$\mathit{\Lambda} + I_1, \quad \mathit{\Lambda} + I_2, \quad \mathit{\Lambda} + G_1, \quad \mathit{\Lambda} + G_2,$$

each of which latter systems, in the old notation, has a lower basic set than (31). What we need, however, is a method for resolving α_3 into its irreducible factors. The irreducibility properties of A_1 and α_2 show that A_1, A_2 is a basic set of a prime system $\mathit{\Lambda}'$ in $u_1, \cdots, u_q;\ y_1, y_2$. Let $R = 0$ be a simple resolvent for $\mathit{\Lambda}'$, constructed as in § 54, with

$$w - a_1 y_1 - a_2 y_2 = 0, \tag{32}$$

a_1, a_2 being integers. It is clear that $a_1 \zeta' + a_2 \zeta''$ annuls R when substituted for w, and that ζ' and ζ'' are each rational in $a_1 \zeta' + a_2 \zeta''$, with coefficients in $\mathfrak{D}_0$. In short, if B is any form in the u_i, y_i and w, which holds $\mathit{\Lambda}'$ and the first member of (32), and if C results from B on replacing w by $a_1 y_1 + a_2 y_2$, then C holds $\mathit{\Lambda}'$. Then

$$I_1^{\mu_1} I_2^{\mu_2} C = K_1 A_1 + K_2 A_2,$$

so that C vanishes for $y_1 = \zeta'$, $y_2 = \zeta''$. Thus, in factoring α_3, we may use the domain of rationality obtained by adjoining

the u_i and $a_1 \zeta' + a_2 \zeta''$ to $\mathfrak{D}$. The factorization is accomplished as in § 50.

All in all, we have a method for testing $\varLambda$ to determine whether (31) is a basic set of a prime system, and for replacing $\varLambda$ by a set of systems each with basic sets lower than (31) when the test is negative.*

Using now the old notation for the unknowns, let us suppose that (31) has been found to be a basic set for a prime system. Let Σ_1 denote the latter system. Then $\varLambda$ is equivalent to

$$\Sigma_1, \ \varLambda + I_1, \cdots, \varLambda + I_p.$$

Each $\varLambda + I_i$ has a basic set which is lower than (31).

What precedes shows that the given system Σ can be resolved into prime systems, as far as the determination of basic sets of the prime systems goes, by a finite number of rational operations and factorizations, provided that the same can be done for all finite systems whose basic sets are lower than those of Σ. The final remark of § 3 gives a quick abstract proof that the resolution is possible for Σ. What is more, the processes used above, of reduction, factorization and isolation of prime systems Σ_1, give an algorithm for the resolution.

56. It remains to solve the following problem: Given a basic set

$$A_1, \cdots, A_p \tag{33}$$

of a non-trivial prime system Ω in $y_1, \cdots, y_n$, each A_i being of class $q + i$, $(p + q = n)$, it is required to find a finite system of forms equivalent to Ω.†

57. Let

$$z_i = t_{i1} y_1 + \cdots + t_{in} y_n, \qquad i = 1, \cdots, n, \tag{34}$$

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* If, when the unknowns are $u_1, \cdots, u_q$; $y_1, \cdots, y_p$, (31) is a basic set for the prime system Ω, then, when we revert to the old notation, (31) will be a basic set for the system into which Ω goes.

† Σ of § 55 leads to several systems Ω. For each Ω, we reletter the unknowns appropriately. After finite systems are found, equivalent to the various Ω, we can revert to the original lettering.

where the z_i, t_{ij} are new unknowns. Given any $q+1$ of the z_i,

$$z_{i_1}, \cdots, z_{i_{q+1}},$$

we find, by the method of § 31, a non-zero form in those z_i and the t_{ij} which vanishes for arbitrary t_{ij}, provided that the z_i are obtained, according to (34), from y_i which satisfy (33) and make no initial in (33) zero.

Let B be such a form in $z_1, \cdots, z_{q+1}$. Let m be the degree of B considered as a polynomial in the z_i. We shall show how to obtain a relation $C=0$ among $z_1, \cdots, z_{q+1}$, where C is of degree m as a polynomial in the z_i and, in addition, is of degree m in each z_i separately, $i=1, \cdots, q+1$.

Let

$$(35) \qquad z_i = a_{i1} z_1' + \cdots + a_{i,q+1} z_{q+1}', \qquad i = 1, \cdots, q+1,$$

where the z_i' and the a_{ij} are new unknowns.

Then $B=0$ goes over into a relation $B'=0$, B' being a polynomial in the z_i' whose coefficients are simple forms in the t_{ij}, a_{ij}. The degree of B' in each z_i' will be effectively m.* Furthermore, we can specialize the a_{ij} as integers, in such a way that the determinant $|a_{ij}|$ is not zero and that the coefficient of the mth power of each z_i' in B' becomes a non-zero simple form in the t_{ij}. Let this be done, and let B'' be the form in the z_i', t_{ij} into which B' thus goes.

From (34), (35), we find

$$(36) \qquad z_i' = \tau_{i1} y_1 + \cdots + \tau_{in} y_n, \qquad i = 1, \cdots, q+1,$$

where each τ_{ij} is a linear combination, with rational numerical coefficients, of the t_{ij} with $i \leqq q+1$. From (35), (36), we see that the t_{ij} with $i \leqq q+1$ are linear combinations of the τ_{ij}, with integral coefficients. Hence, the τ_{ij} may be made to become arbitrarily assigned analytic functions, if the t_{ij} are taken appropriately.

In the relation $B''=0$, we substitute, for each t_{ij}, its expression in terms of the τ_{ij}. Then B'' goes over into a

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* Perron, *Algebra*, vol. 1, p. 288.

form B''' in the z'_i, τ_{ij}, $i = 1, \cdots, q+1$. We now replace, in B''', each τ_{ij} by t_{ij} and each z'_i by z_i. Then B''' goes over into a form C in $z_1, \cdots, z_{q+1}$ and the t_{ij}, C being of degree m as a polynomial in the z_i, and of degree m in each z_i separately. Furthermore C vanishes for all z_i given by (34) for which the y_i satisfy (33) and make no initial zero. This is because (36) may be considered equivalent to the first $q+1$ relations (34).

Evidently the relation $C = 0$ will subsist if we replace $z_1, \cdots, z_{q+1}$ by any $q+1$ of the z_i, provided that a corresponding substitution is made for the t_{ij} in C.

We now specialize the t_{ij} in (34) as integers with a non-vanishing determinant, in such a way that the relations obtained from $C = 0$ for the various sets of $q+1$ unknowns remain of effective degree m in each z_i appearing in them. These relations will have coefficients in $\mathfrak{D}$.

58. We consider $z_1, \cdots, z_n$ with the t_{ij} fixed as above. If the y_i are replaced in (33) in terms of the z_i, we get a system Φ of p forms in the z_i. Let basic sets be determined for a set of prime systems equivalent to Φ. Let $\Sigma_1, \cdots, \Sigma_s$ be those prime systems which are not held by the initial of any A_i in (33), the y_i being replaced in the initials in terms of the z_i.* There will be one of the Σ_i which holds the remaining Σ_i. This is because, in a resolution of (33) into indecomposable systems none of which holds any other, there is precisely one which is held by no initial. To determine that Σ_i which holds the others, all we need do is to find a Σ_i whose basic set holds the other Σ_i. Suppose, for instance, that the basic set of Σ_1 holds $\Sigma_2, \cdots, \Sigma_s$. Then, if Σ_1 does not hold Σ_j, the initial of some form in the basic set of Σ_1 must hold Σ_j. Then surely Σ_j cannot hold Σ_1. Thus if Σ_1 does not hold all Σ_i, no Σ_j can hold all Σ_i. Then Σ_1 holds all Σ_i.

Σ_1 is obtained from Ω, (§ 56) by replacing the y_i in terms of the z_i. We shall prove that Σ_1, like Ω, has q unconditioned

* The condition for a form to hold a prime system is that its remainder with respect to the basic set vanish.

unknowns (§ 42). To begin with, it is easy to see that the forms in any $q+1$ of the z_i, found in § 57, belong to Σ_1. On the other hand, if there were fewer than q unconditioned unknowns in Σ_1, we could use the basic set of Σ_1 to determine a non-zero form in $y_1, \cdots, y_q$ belonging to Ω.

Changing the notation if necessary, let $z_1, \cdots, z_q$ be unconditioned unknowns for Σ_1. Then Σ_1 will have a basic set

$$B_1, \cdots, B_p \tag{37}$$

in which B_i introduces z_{q+i}. We assume, as we may, that B_1 is algebraically irreducible.

59. We construct a simple resolvent $R=0$ for Σ_1, with

$$w = a_1 z_{q+1} + \cdots + a_p z_{q+p}, \tag{38}$$

the a_i being integers. Let R be of degree g in w.

We shall prove that the initial of R is a function of x in $\mathfrak{D}$. According to § 57, each z_i, $i>q$, satisfies with $z_1, \cdots, z_q$ an equation of degree m in z_i, the coefficient of z_i^m being a function of x in $\mathfrak{D}$. We may and shall assume that the coefficient of z_i^m, $i>q$, in each of these p equations is unity. Then (38) shows that w satisfies with $z_1, \cdots, z_q$ an equation in which the coefficient of the highest power of w is unity.* This implies that, in the algebraically irreducible simple form R, the coefficient of w^g is free of $z_1, \cdots, z_q$. We may and shall assume that the coefficient of w^g is unity.

We shall show that

$$z_i = \frac{E_{i0} + E_{i1} w + \cdots + E_{i,g-1} w^{g-1}}{D} \tag{39}$$

$i=q+1, \cdots, n$, where the E_{ij} and D are forms in $z_1, \cdots, z_q$. Let

$$M z_{q+1} - N = 0,$$

where M and N are forms in w; $z_1, \cdots, z_q$ of degree less than g in w. As M and R are relatively prime polynomials, we have

* This is analogous to the fact that the sum of several algebraic integers is an algebraic integer. See Landau, *Zahlentheorie*, vol. 3, p. 71.

$$PM+QR = L$$

where L is a non-zero form in $z_1, \cdots, z_q$. Then

$$Lz_{q+1} - PN = 0,$$

and, replacing PN by its remainder with respect to R, we have a relation (39) for z_{q+1}. Evidently we may use the same D for $z_{q+1}, \cdots, z_n$.

60. Let $u_1, \cdots, u_p$; v be new unknowns and let $\mathcal{A}$ be the totality of simple forms in the z_i, u_i and v which hold Σ_1 and

$$v - u_1 z_{q+1} - \cdots - u_p z_n.$$

Then $\mathcal{A}$ has an algebraically irreducible form Z in $v, z_1, \cdots, z_q$ and the u_i, the coefficient of whose highest power of v is unity.*

We shall prove that Z is of degree g in v. Using (39), we see that

$$v = \frac{K_0 + \cdots + K_{g-1} w^{g-1}}{D}$$

where the K_i are simple forms in $z_1, \cdots, z_q$ and the u_i, and where w is given by (38). For the first g powers of v, we get similar expressions if we make use of $R = 0$. We infer, by a linear dependence argument, that v satisfies, with $z_1, \cdots, z_q$ and the u_i, if $D \neq 0$, an equation of degree at most g in v. The condition that $D \neq 0$ is removed by considering that $\mathcal{A}$ is prime. Thus Z is at most of degree g in v. On the other hand, as v becomes w if $u_i = a_i$, $i = 1, \cdots, p$, Z cannot be of degree less than g in v.†

Let v be replaced in Z by

$$(40) \qquad u_1 z_{q+1} + \cdots + u_p z_n.$$

Then Z becomes a form in $z_1, \cdots, z_n$ and the u_i. Let this form be arranged as a polynomial in the u_i with coefficients which are forms in the z_i.

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* Note that each $u_i z_{q+i}$ satisfies an equation with the coefficient of the highest power of $u_i z_{q+i}$ equal to unity.

† Note that, since the coefficient of the highest power of v is free of the u_i, Z cannot vanish identically for $u_i = a_i$.

Let Ψ be the finite system of these coefficients (forms in the z_i). We are going to prove, in the following sections, that Ψ is equivalent to Σ_1. Thus, if the z_i are replaced in Ψ by their expressions (34), we get a finite system of forms equivalent to Ω. We shall thus have solved the problem stated in § 56.

61. We begin with the observation that for given analytic functions $z_1, \cdots, z_n$ to constitute a solution of Ψ, it is necessary and sufficient that, for v as in (40), and for $z_1, \cdots, z_q$ as just given, Z vanish for arbitrary u_i. This shows, in particular, that Ψ holds Σ_1.

Let G be the discriminant of R with respect to w and let

$$K = DG,$$

where D is as in (39). We shall prove that every solution of Ψ with $K \neq 0$ is a solution of Σ_1. Let $\xi_1, \cdots, \xi_n$ be such a solution of Ψ. Corresponding to $\xi_1, \cdots, \xi_q$, the equation $R = 0$ gives g distinct solutions for w. Using each such w in (39), we get g distinct solutions

$$\xi_1, \cdots, \xi_q, z_{q+1}^{(j)}, \cdots, z_n^{(j)}, \quad j = 1, \cdots, g$$

of Σ_1. Let β be the polynomial which Z becomes for $z_i = \xi_i$, $i = 1, \cdots, q$. Then

$$\beta = \prod_{j=1}^{g} (v - u_1 z_{q+1}^{(j)} - \cdots - u_p z_n^{(j)}).$$

But $v - u_1 \xi_{q+1} - \cdots - u_p \xi_n$ is a factor of β. This shows that for some j, $z_i^{(j)} = \xi_i$, $i = q+1, \cdots, n$, and proves our statement.

62. We are going to show that, given any solution $\xi_1, \cdots, \xi_n$ of Ψ, the ξ_i being analytic in some open region $\mathfrak{A}_1$ in $\mathfrak{A}$, there exists an open region $\mathfrak{A}'$, contained in $\mathfrak{A}_1$, in which the solution can be approximated uniformly by a solution of Ψ for which, throughout $\mathfrak{A}'$, $K \neq 0$. That is, for every $\varepsilon > 0$, there exists a solution $\eta_1, \cdots, \eta_n$ of Ψ, analytic throughout $\mathfrak{A}'$, such that $K \neq 0$ throughout $\mathfrak{A}'$ and that $|\xi_i - \eta_i| < \varepsilon$ throughout $\mathfrak{A}'$, $i = 1, \cdots, n$.

This will show that Σ_1 holds Ψ, for since a form in Σ_1 vanishes for every solution of Ψ with $K \neq 0$, it will vanish, by continuity, for every solution of Ψ. We shall thus know that Ψ and Σ_1 are equivalent.

63. We shall establish the more general result that if H is any non-zero simple form in $z_1, \cdots, z_q$, then given any solution of Ψ analytic in $\mathfrak{A}_1$, there is an $\mathfrak{A}'$ in $\mathfrak{A}_1$ in which the solution can be approximated, as above, by a solution of Ψ with H distinct from zero throughout $\mathfrak{A}'$.

It will evidently suffice to consider a solution of Ψ for which $H = 0$.

We assume $\mathfrak{A}_1$ to be so taken that the equations of degree m which $z_{q+1}, \cdots, z_n$ each satisfy with $z_1, \cdots, z_q$ (§ 59) have their coefficients analytic throughout $\mathfrak{A}_1$. We assume also that the coefficients in H, in K of § 61, in R of § 59 and in D and the E_{ij} of (39) are analytic throughout $\mathfrak{A}_1$.

There is evidently no loss of generality in assuming that H is divisible by K. We make this assumption.

Let $b_1, \cdots, b_q$ be constants such that

$$H(\xi_1 + b_1, \cdots, \xi_q + b_q)$$

does not vanish for every x. Then, if h is a complex variable,

$$H(\xi_1 + b_1 h, \cdots, \xi_q + b_q h) \tag{41}$$

is a polynomial in h of the type

$$\alpha_r h^r + \cdots + \alpha_s h^s, \tag{42}$$

where the α_i are functions of x analytic in $\mathfrak{A}_1$. Since (42) vanishes for $h = 0$, we have $r \geqq 1$. We assume that α_r is not zero for every x.

Let $\mathfrak{A}_2$ be an open region in $\mathfrak{A}_1$ in which α_r is bounded away from zero. Let h be small, but distinct from zero. Then (42) cannot be zero at any point of $\mathfrak{A}_2$. Thus, if

$$z_i = \xi_i + b_i h, \qquad i = 1, \cdots, q, \tag{43}$$

$R = 0$ will have g distinct solutions for w, each analytic in $\mathfrak{A}_2$. This is because H is divisible by the discriminant of R.

As H is divisible by D in (39), $|\Sigma_1$ will have g distinct solutions with $z_1, \cdots, z_q$ as in (43), for which $z_{q+1}, \cdots, z_n$ are given by (39) and are analytic throughout $\mathfrak{A}_2$.

Consider a sequence of non-zero values of h which tend towards zero,

$$h_1, h_2, \cdots, h_i, \cdots \tag{44}$$

each h_i being so small that (42) is distinct from zero throughout $\mathfrak{A}_2$. For each h_i, if

$$z_j = \xi_j + b_j h_i, \qquad j = 1, \cdots, q, \tag{45}$$

Z will vanish if

$$v = u_1 z_{q+1}^{(k)} + \cdots + u_p z_n^{(k)}, \tag{46}$$

$k = 1, \cdots, g$, where the $z_j^{(k)}$ are analytic throughout $\mathfrak{A}_2$. It is understood, of course, that the $z_j^{(k)}$ depend on h_i. For any h_i, the g expressions (46) are distinct from one another.

As the equation of degree m which each z_j, $j > q$ satisfies with $z_1, \cdots, z_q$ has unity for the coefficient of z_j^m, (§ 59), there is a region $\mathfrak{A}_3$ in $\mathfrak{A}_2$ and a positive number d such that, throughout $\mathfrak{A}_3$,

$$|z_j^{(k)}| < d \tag{47}$$

for $j = q+1, \cdots, n$; $k = 1, \cdots, g$ and for every h_i in (44). This is because the coefficients of $z_j^{m-1}, \cdots, z_j^0$, in the above considered equation, are bounded quantities.

For each h_i of (44), let one of the g expressions (46) be selected, and designated by $v^{(i)}$. We form thus a sequence

$$v', v'', \cdots, v^{(i)}, \cdots. \tag{48}$$

Let $\mathfrak{A}'$ be any bounded open region which lies, with its boundary, in $\mathfrak{A}_3$. From (47) we see, using a well known theorem on bounded sequences of analytic functions,* that, for some subsequence of (48), the coefficients of each u_i, $i = 1, \cdots, p$, converge uniformly throughout $\mathfrak{A}'$ to an analytic function. Let the limit, for the subsequence, of the coefficient of u_i be ζ_i'. We find thus that if

* Montel, *Les familles normales de fonctions analytiques*, p. 21. Dienes, *The Taylor Series*, p. 160.

(49) $$z_j = \xi_j, \qquad j = 1, \cdots, q,$$
Z vanishes for
$$v = u_1 \zeta'_{q+1} + \cdots + u_p \zeta'_n.$$

Deleting elements of (44) if necessary, we assume that the convergence occurs when the complete sequence (48) is used, rather than one of its subsequences. For each h_i there are $g-1$ expressions (46) not used in (48). Let one of these $g-1$ expressions be selected for each h_i, and let (48) be used now to represent the sequence thus obtained. As above, we select a subsequence of (48) for which the coefficients of each u_i converge uniformly in $\mathfrak{A}'$. This gives a second expression
$$v = u_1 \zeta''_{q+1} + \cdots + u_p \zeta''_n$$
which causes Z to vanish when (49) holds. Continuing, we find g expressions

(50) $$v = u_1 \zeta^{(k)}_{q+1} + \cdots + u_p \zeta^{(k)}_n,$$

$k = 1, \cdots, g$, which make Z vanish when (49) holds.

Let v_k represent the second member of (50). Again, let w_k represent the second member of (46), it being understood that the subscripts k are now assigned, for each h_i, in such a way that the coefficient of u_i in w_k converges to that in v_k as h_i approaches 0.

Then since the g expressions w_k are distinct from one another for every h_i, we will have, representing by β the polynomial which Z becomes when (45) holds,
$$\beta = (v - w_1) \cdots (v - w_g).$$
By continuity, if we represent Z, when (49) holds, by γ,
$$\gamma = (v - v_1) \cdots (v - v_g).$$
But since $\xi_1, \cdots, \xi_n$ is a solution of Ψ,
$$v - u_1 \xi_{q+1} - \cdots - u_p \xi_n$$
must be a factor of γ. This shows that, for some k,
$$\xi_i = \zeta^{(k)}_i, \qquad i = q+1, \cdots, n.$$

This establishes the result stated at the head of the present section and proves that Ψ is equivalent to Σ_1.

A SPECIAL THEOREM

64. We prove the following theorem.

THEOREM: *Let Σ be an indecomposable system of simple forms in $y_1, \cdots, y_n$. Let B be any simple form which does not hold Σ. Given any solution of Σ, analytic in an open region $\mathfrak{A}_1$, there is an open region $\mathfrak{A}'$, contained in $\mathfrak{A}_1$, in which the given solution can be approximated uniformly, with arbitrary closeness, by solutions of Σ for which B is distinct from 0 throughout $\mathfrak{A}'$.*

We assume, without loss of generality that Σ is prime. If the transformation of § 57 is effected, Σ may be replaced by Σ_1, while B goes over into a form C in $z_1, \cdots, z_n$.

C does not hold Σ_1. Let $z_{q+1}, \cdots, z_n$ be replaced in C by their expressions (39). We find that, for all solutions of Σ_1, with $D \neq 0$,

$$C = \frac{N}{D^\mu}, \tag{51}$$

where N is a simple form in w; $z_1, \cdots, z_q$. In (51), w is supposed to be given by the second member of (38). Because DC does not hold Σ_1, N is not divisible by R of § 59. Thus, we have

$$XR + YN = H, \tag{52}$$

where H is a non-zero simple form in $z_1, \cdots, z_q$.

Let $\mathfrak{A}_2$ be a region, contained in $\mathfrak{A}_1$, in which the coefficients of the forms in (51) and (52) are analytic. We see that if a solution of Σ_1 makes C vanish at some point c in $\mathfrak{A}_2$, then DH vanishes at c. But there is a region $\mathfrak{A}'$ in $\mathfrak{A}_2$ in which any given solution of Σ_1 can be approximated uniformly by a solution for which DH, hence C, is distinct from zero throughout $\mathfrak{A}'$. As the y_i vary continuously with the z_i, we have our theorem.*

* This useful theorem, and the considerations which lead up to it, do not seem to exist in the literature, even for the case of equations with constant coefficients. Professor van der Waerden recently communicated to me a different proof, which deals with the case of constant coefficients.

CHAPTER V

CONSTRUCTIVE METHODS

CHARACTERIZATION OF BASIC SETS OF IRREDUCIBLE SYSTEMS

65. Let

$$(1)\qquad A_1, A_2, \cdots, A_p$$

be an ascending set of differential forms in

$$u_1, \cdots, u_q; \qquad y_1, \cdots, y_p,$$

each A_i being of class $q+i$. We are going to find a necessary and sufficient condition for (1) to be a basic set of a closed irreducible system.

Let the order of A_i in y_i be r_i. We represent y_{ir_i} by z_i, $i = 1, \cdots, p$. The remaining y_{ij} in (1) and the u_{ij} present in (1), we designate now by symbols v_k, attributing the subscripts k in any arbitrary manner. With these replacements of symbols, (1) goes over into an ascending set of simple forms,

$$(2)\qquad B_1, B_2, \cdots, B_p$$

in the unknowns

$$(3)\qquad v_1, \cdots, v_r; \; z_1, \cdots, z_p.$$

The passage from (1) to (2) is purely formal. Once it is effected, we treat (2) like any other set of simple forms in the v_i, z_i. For instance, whereas, in a solution of (1), $y_{i,j+1}$ must be the derivative of y_{ij}, any set of analytic v_i, z_i which annul the B_i will be considered a solution of (2).

We are going to prove that *for* (1) *to be a basic set of a closed irreducible system, it is necessary and sufficient that* (2) *be a basic set for a prime system in the unknowns* (3), *the domain of rationality being* $\mathfrak{F}$.

We prove first the necessity. Suppose that the condition is not fulfilled. Referring to § 55, and also §§ 46, 47, we see that, since (2) is not a basic set of a prime system, there exists, for some j, an identity

$$I_1^{\mu_1} \cdots I_{j-1}^{\mu_{j-1}} (TB_j - G_1 G_2) - K_1 B_1 - \cdots - K_{j-1} B_{j-1} = 0,$$

where I_i is the initial of B_i and G_1, G_2 are non-zero forms in the unknowns in $B_1, \cdots, B_j$, which are reduced with respect to $B_1, \cdots, B_j$.

To this identity, there corresponds an identity in forms in the u_i, y_i,

$$(4)\quad J_1^{\mu_1} \cdots J_{j-1}^{\mu_{j-1}} (SA_j - H_1 H_2) - L_1 A_1 - \cdots - L_{j-1} A_{j-1} = 0,$$

where J_i is the initial of A_i. Here H_1 and H_2 are non-zero forms of class $q+j$, which are reduced with respect to $A_1, \cdots, A_j$. Thus, if (1) were a basic set of a closed irreducible system, either H_1, H_2 or some J_i would belong to the system. This completes the necessity proof.

Suppose now that the condition is fulfilled. We shall prove first that (1) has regular solutions. Consider any regular solution of (2). Let a be a value of x for which the functions in the solution, and the coefficients in (2), are analytic, and for which no initial or separant in (2) is zero (for the given solution). Let the values of the v_i, z_i at a be assigned to the corresponding u_{ij}, y_{ij} in (1). We construct functions $u_1, \cdots, u_q$, analytic at a, for which the u_{ij} in (1) have the indicated numerical values. If we assign to the first $r_1 - 1$ derivatives of y_1, at a, the numerical values associated with them above, the differential equation $A_1 = 0$ will have a regular solution in which the u_i are the functions above and in which the first r_1 derivatives of y_1 have, at a, the above assigned values. We substitute $u_1, \cdots, u_q$; y_1 into A_2 and solve for y_2 with the initial conditions determined above. We have now a regular solution of A_1, A_2. Continuing, we find a regular solution of (1).

Now, let G and H be two forms such that GH vanishes for all regular solutions of (1). Let G_1 be the remainder

of G with respect to (1), and H_1 the remainder of H. Then $G_1 H_1$ vanishes for all regular solutions of (1). It may be that G_1 and H_1 involve certain u_{ij} not effectively present in (1). In that case, let new symbols v_i be added to (3) for the new u_{ij}. Then (2) will be a basic set for a prime system even after this adjunction of unknowns, for it will continue to satisfy the condition of § 45.

As we saw above, the values of the functions in a regular solution of (2), at a point a which is quite arbitrary, are values of the u_{ij}, y_{ij} in a regular solution of (1). This means, if G_2 and H_2 are obtained from G_1 and H_1 by replacing the u_{ij}, y_{ij} by the v_i, z_i, that $G_2 H_2$ vanishes for all regular solutions of (2). Hence either G_2 vanishes for all regular solutions of (2) or H_2 does. Suppose that G_2 does. As G_2 is reduced with respect to (2), G_2 vanishes identically. Then G vanishes for every regular solution of (1).

Thus, the totality Σ of forms which vanish for the regular solutions of (1) is an irreducible system. What precedes shows that if a form G belongs to Σ, the remainder of G with respect to (1) is zero. This means that Σ has no non-zero form reduced with respect to (1), so that (1) is a basic set of Σ. The sufficiency proof is completed.

66. We shall prove that *if* (1) *is a basic set of a closed irreducible system* Σ, *then every solution of* (1) *for which no separant vanishes is a solution of* Σ.

Let S_i be the separant of A_i. Let G be a form which vanishes for all regular solutions of (1). As in § 5, we can show the existence of integers $s_1, \cdots, s_p$ such that, when a suitable linear combination of derivatives of $A_1, \cdots, A_p$, with forms for coefficients, is subtracted from

$$S_1^{s_1} \cdots S_p^{s_p} G,$$

the remainder, G_1, is not of higher order than any A_i in y_i, $i = 1, \cdots, p$.

Let H result from G_1 when we pass to the unknowns (3). Then H vanishes for every regular solution of (2). Hence, by § 49, H vanishes for every solution of (2) for which no

separant vanishes. Then G_1 vanishes for every solution of (1) for which no separant vanishes. So does G. This proves our statement.

BASIC SETS IN A RESOLUTION OF A FINITE SYSTEM INTO IRREDUCIBLE SYSTEMS

67. Let Σ be any *finite* system of forms in $y_1, \cdots, y_n$, not all zero. In this section, we show how to determine basic sets of a finite number of closed irreducible systems which form a set of systems equivalent to Σ. In Chapter VII, we give a theoretical process for determining finite systems equivalent to the closed irreducible systems.

Let

$$(5) \qquad A_1, A_2, \cdots, A_p$$

be a basic set of Σ, determined as in § 4. If A_1, is of class zero, Σ has no solutions, and is thus irreducible. We assume now that A_1 is not of class zero. For every form in Σ, let the remainder with respect to (5) be determined. If these remainders are adjoined to Σ, we get a system Σ' equivalent to Σ. By § 4, if not all remainders are zero, Σ' will have a basic set of lower rank than (4). We see, by § 3, that after a finite number of repetitions of the above operation, we arrive at a finite system $\varLambda$, equivalent to Σ, with a basic set (5) for which either A_1 is of class zero or for which, otherwise, the remainder of every form in $\varLambda$ is zero.

Let us suppose that we are in the latter case. We shall make a temporary relettering of the y_i. If, in the set (5) for $\varLambda$, A_i is of class j_i, we replace the symbol y_{j_i} by y_i. The $q = n-p$ unknowns not among the y_{j_i}, we call, in any order, $u_1, \cdots, u_q$. We list all the unknowns in the order $u_1, \cdots, u_q; y_1, \cdots, y_p$.

With this change of notation, we determine, by § 65, whether (5) is a basic set of a closed irreducible system. If it is not, we see from (4), that $\varLambda$ is equivalent to

$$\varLambda+J_1, \cdots, \varLambda+J_{j-1}, \qquad \varLambda+H_1, \qquad \varLambda+H_2.$$

Each of the latter systems, after we revert to the old notation, will have a basic set lower than (5).

If when the unknowns are the u_i, y_i, (5) is a basic set of a closed irreducible system Ω, then, when we revert to the old notation, (5) will be a basic set for the closed irreducible system into which Ω goes.

Using now the old notation for the unknowns, let us suppose that (5) has been found to be a basic set for a closed irreducible system. Let Σ_1 denote the latter system. Then, by § 66, Λ is equivalent to

$$\Sigma_1, \ \Lambda + S_1, \ \cdots, \ \Lambda + S_p.$$

Each $\Lambda + S_i$ has a basic set which is lower than (5).

What precedes shows that the given system Σ can be resolved into irreducible systems, as far as the determination of basic sets of the irreducible systems goes, by a finite number of rational operations, differentiations and factorizations, provided that the same can be done for all finite systems whose basic sets are lower than those of Σ. The final remark of § 3 gives an abstract proof that the resolution is possible for Σ. What is more, the processes used above give an algorithm for the resolution.

In the resolution into irreducible systems obtained above, some systems may be held by others.

The algorithm obtained above contains in itself a complete elimination theory for systems of algebraic differential equations. We get all of the solutions of Σ by finding the solutions of each basic set which cause no separant to vanish. A solution of an irreducible system which annuls some separant will be a solution of some system like $\Lambda + S_i$ above, and hence will ultimately be found among the solutions of some other irreducible system, where it annuls no separant. Thus our algorithm reduces the process of determining all solutions of a system of algebraic differential equations to an application of the implicit function theorem and of the existence theorem for differential equations.

It follows from what precedes that *a system of forms in $y_1, \cdots, y_n$, in which each form is linear in the y_{ij}, is an irreducible system.*

Test for a form to hold a finite system

68. Let Φ be any finite system of forms. Let it be required to determine whether a given form G holds Φ. What one does is to resolve Φ into irreducible systems as in § 67. For G to hold Φ, it is necessary and sufficient that G hold each irreducible system. The condition for G to hold one of the irreducible systems is that its remainder with respect to the basic set of the irreducible system be zero. This gives a test which involves a finite number of steps.

Construction of resolvents

69. Let

$$A_1, A_2, \cdots, A_p, \tag{6}$$

where the A_i are forms in $u_1, \cdots, u_q$; $y_1, \cdots, y_p$, each A_i of class $q+i$, be given as a basic set of a closed irreducible system Σ. We suppose that either $\mathfrak{F}$ does not consist entirely of constants, or u_i actually exist.

We shall show how to construct a resolvent for Σ.

We begin by showing how to obtain the form G of § 25. Let B_i be the form obtained from A_i, by replacing each y_j by a new unknown z_j. We consider the finite system Ω composed of the forms of (6), the forms

$$B_1, \cdots, B_p \tag{7}$$

and also

$$\lambda_1(y_1 - z_1) + \cdots + \lambda_p(y_p - z_p),$$

where the λ_i are unknowns. We order the unknowns as follows:

$$u_1, \cdots, u_q; \quad \lambda_1, \cdots, \lambda_p; \quad y_1, \cdots, y_p; \quad z_1, \cdots, z_p.$$

We apply the process of § 67 for resolving Ω into irreducible systems, each irreducible system being represented by a basic set. The theory of §§ 25, 26 shows that each irreducible system which is not held by every form $y_i - z_i$ has a basic set containing a form in the u_i and λ_i alone. We obtain, by a multiplication of such forms, the form K of § 25.

When $\mathfrak{F}$ contains a non-constant function, the determination of μ_i which do not annul L of § 25 is an elementary problem

whose solution is sufficiently indicated in § 25. When u_i exist, we find the M_i of § 26 by inspection.

To avoid tedious discussions of notation, let us limit ourselves now to the case in which $\mathfrak{F}$ does not consist of constants. Consider the system

$$(8) \qquad A_1, \cdots, A_p, \quad w-(\mu_1 y_1+\cdots+\mu_p y_p)$$

in the unknowns

$$(9) \qquad u_1, \cdots, u_q; \; y_1, \cdots, y_p; \; w.$$

The totality of forms which vanish for all solutions of (8) which annul no separant is the system Ω of § 28. Every other closed essential irreducible system held by (8) is held by some separant.

We rearrange the unknowns (9) in the order

$$u_1, \cdots, u_q; \; w; \; y_1, \cdots, y_p,$$

and apply the process of § 67 to resolve (8) into irreducible systems. We test these irreducible systems to see whether they are held by the separant of some A_i, and pick out those, say $\Sigma_1, \cdots, \Sigma_s$, which are held by no separant.

As (8) has only one essential irreducible system which is held by no separant, there must be one Σ_i which holds all other Σ_i. To find such a Σ_i, we need only find a Σ_i whose basic set holds all other Σ_i. For, let the basic set of Σ_1, hold $\Sigma_2, \cdots, \Sigma_s$. If Σ_1 does not hold Σ_j, the separant of some form in the basic set of Σ_1, must hold Σ_j, so that Σ_j cannot hold Σ_1. Thus, if Σ_1 does not hold every Σ_i, no Σ_j can hold every Σ_i.

Σ_1 will have a basic set

$$R, R_1, \cdots, R_p$$

in which R is an algebraically irreducible form in w and the u_i and in which R_i, $i=1, \cdots, p$, introduces y_i. By §§ 28, 29 each equation $R_i=0$ determines y_i rationally in w and the u_i and $R=0$ is a resolvent for Σ.

A REMARK ON THE FUNDAMENTAL THEOREM

70. The results of §§ 65–67 furnish a new proof of the fact that every *finite* system of forms is equivalent to a finite number of irreducible *infinite* systems. Using the lemma of § 7, we obtain the theorem of § 13. This new proof of the fundamental theorem appears to us not to depend on Zermelo's axiom. But only that part which is stated above has been demonstrated on a genuinely constructive basis.

JACOBI-WEIERSTRASS CANONICAL FORM

71. Let Σ be a closed irreducible system with (1) for basic set. Let Λ be the prime system for which (2) is a basic set. We build a simple resolvent, $R = 0$, for Λ, with

$$w - a_1 z_1 - \cdots - a_p z_p = 0,$$

the a_i being integers. We have

$$M_i z_i - N_i = 0 \qquad i = 1, \cdots, p, \tag{10}$$

where the M_i, N_i are simple forms in w and the v_i.

Let Ω be the totality of forms which vanish for the common solutions of Σ and

$$w - a_1 y_{1r_1} - \cdots - a_p y_{pr_p}.$$

Let R go over into a form R' when the v_i, z_i, are replaced by the corresponding u_{ij}, y_{ij}. Similarly, let (10) go over into

$$M_i' y_{ir_i} - N_i' = 0. \tag{11}$$

Then R' and the first members of (11) are in Ω. It can be shown that Σ consists of all forms in the u_i, y_i which vanish for all solutions of (11) and $R' = 0$ for which the separant of R' and the M_i' do not vanish.

Suppose that there are no u_i. In that case, the system (11), with w defined by $R' = 0$, when converted into a system of the first order, by the method of adjunction of unknowns used in differential equation theory, assumes a form equivalent to the Jacobi-Weierstrass canonical form.*

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* Forsythe, *Treatise on Differential Equations*, vol. II, pp. 11-14.

CHAPTER VI

CONSTITUTION OF AN IRREDUCIBLE MANIFOLD

SEMINORMAL SOLUTIONS

72. Let Σ be a non-trivial closed irreducible system in $u_1, \cdots, u_q; y_1, \cdots, y_p$ for which

$$(1) \qquad A_1, A_2, \cdots, A_p,$$

each A_i of class $q+i$, is a basic set.

A solution of (1) for which no separant vanishes will be called a *normal* solution of (1). By § 66, *every normal solution of* (1) *is a solution of* Σ.

A solution

$$(2) \qquad \bar{u}_1, \cdots, \bar{u}_q; \quad \bar{y}_1, \cdots, \bar{y}_p$$

of (1) for which some separant vanishes will be called *seminormal* if there exists a set of points, dense in the area $\mathfrak{B}$ in which the functions in (2) are analytic, such that, given any point a of the set, any positive integer m, and any $\epsilon > 0$, there exists a normal solution of (1), $u_1, \cdots, y_p$, analytic at a, such that

$$(3) \qquad \begin{gathered} |u_{ik}(a) - \bar{u}_{ik}(a)| < \epsilon, \qquad |y_{jk}(a) - \bar{y}_{jk}(a)| < \epsilon, \\ i = 1, \cdots, q; \quad j = 1, \cdots, p; \quad k = 0, \cdots, m. \end{gathered}$$

The results of this section and of § 73 will show that the existence of a single point a, as above, implies the existence of a set of such points dense in $\mathfrak{B}$. That is, a solution for which some separant vanishes, and for which a single point a exists, is a seminormal solution.

We shall prove that *if G is a form with coefficients meromorphic in $\mathfrak{A}$, the coefficients not belonging necessarily to $\mathfrak{F}$,*

and if G vanishes for all normal solutions of (1), *then G vanishes for all seminormal solutions of* (1).

More generally, we shall show that G vanishes for every solution (2) for which a single point a, as above, exists. Multiplying G by a power of $(x-a)$, if necessary, we assume that the coefficients in G are analytic at a. When (2) is substituted into G, G becomes a function $\varphi(x)$ of x which is zero at a. This is because G vanishes for all normal solutions and because of the m, ϵ property of a. Similarly, $\varphi'(x)$ vanishes at a, because the derivative of G vanishes for every normal solution. In the same way, every derivative of $\varphi(x)$ vanishes at a, so that $\varphi(x)$ is identically zero, and G vanishes for (2).

If we restrict ourselves to forms G with coefficients in $\mathfrak{F}$, we see that *every seminormal solution of* (1) *is a solution of* Σ.

73. We are going to prove that *the manifold of Σ is composed of the normal solutions of* (1) *and of the seminormal solutions.*

In particular, *the general solution of an algebraically irreducible form A is composed of the normal solutions of A and of the seminormal solutions.*

Let A_i be of order r_i in y_i. Let S_i be the separant of A_i. For every y_{is} with $s > r_i$, in a normal solution of (1), we have an expression

$$y_{is} = \frac{B}{F} \tag{4}$$

where B is a form of class at most $q+i$ and of order at most r_j in y_j, $j = 1, \cdots, i$, and where F is a product of powers of $S_1, \cdots, S_i$. The forms

$$F y_{is} - B \tag{5}$$

belong to Σ.

Let m be any integer greater than every r_i. We adjoin to (1) all forms (5), for $i = 1, \cdots, p$, with $s \leqq m$. Without going through the formality of replacing the u_{ij}, y_{ij} by new symbols, let us consider the forms in (1) and (5) as a system Φ of simple forms in the u_{ij}, y_{ij}. That is any set of analytic

functions u_{ij}, y_{ij}, which annul the forms of Φ will be a solution of Φ. We do not ask, for instance, that $y_{i,j+1}$ be the derivative of y_{ij}.

We shall prove that the totality Ω of simple forms which vanish for all solutions of Φ for which no S_i vanishes, is a prime system. Let GH vanish for all solutions of Φ which annul no S_i. By (5), we have, for these solutions,

$$G = \frac{B_1}{F_1}, \qquad H = \frac{B_2}{F_2}$$

where B_1 and B_2 involve no y_{ij} with $j > r_i$ and where F_1 and F_2 are power products of the S_i. Then $B_1 B_2$ vanishes for the above solutions.

By § 65, (1), regarded as a set of simple forms, is the basic set of a prime system (even after new u_{ij} are introduced). Then either B_1 vanishes for all solutions of the simple forms (1) which annul no S_i or B_2 does. Suppose that B_1 does. Then G vanishes for all solutions of Φ which annul no separant so that Ω is prime.

We shall prove that, given any solution of Σ, the u_{ij}, y_{ij} appearing in Φ, obtained from the solution, constitute a solution of Ω. This is obvious for the normal solutions of (1). Then if G is a form in Ω, G, considered as a differential form in the u_i, y_j, holds Σ. This proves our statement.

Now let (2) be a solution of Σ which annuls some S_i. Consider the corresponding solution of Ω. By § 64, there is a region $\mathfrak{A}'$ such that, given any $\varepsilon > 0$, we can find a solution u_{ik}, y_{jk} of Ω, with no S_i zero at any point of $\mathfrak{A}'$, such that (3) holds at every point a of $\mathfrak{A}'$. We suppose $\mathfrak{A}'$ to be taken so that the coefficients in (1) are analytic throughout $\mathfrak{A}'$.

Now if a is any point of $\mathfrak{A}'$, the $u_{ik}(a)$, $y_{jk}(a)$ in (3) furnish initial conditions for a normal solution of the basic set of differential forms (1) (§ 65). Thus for any a in $\mathfrak{A}'$, there exists a normal solution u_i, y_j of (1) which satisfies (3) with the solution (2).

We repeat the above procedure, using $2m$ and $\epsilon/2$ in place of m and ϵ. We find a region $\mathfrak{A}''$, in $\mathfrak{A}'$, every point a of which can be used as above. Employing $4m$ and $\epsilon/4$, we find a region $\mathfrak{A}'''$ in $\mathfrak{A}''$. We continue, determining a sequence of regions $\mathfrak{A}^{(i)}$. There is at least one point a common to all of these regions. Given any $\epsilon > 0$, and any m, there is a normal solution of (1), analytic at a, for which (3) holds. As there is an a in every area in which (2) is analytic, (2) is a seminormal solution of (1). Our result is proved.

It is very likely that the set of points a consists of all points at which the functions in (2) are analytic, with the possible exception of an isolated set. One might ask, also, whether every seminormal solution can be approximated uniformly in some area, with arbitrary closeness, by a normal solution. A negative answer would certainly be interesting. These questions need more attention than we have been able to give them.

Example. Consider the form in the unknown y,

$$A = (y\,y_2 - y_1^2)^2 - 4y\,y_1^3.$$

It is algebraically irreducible in the field of all constants because, when equated to zero, it defines y_2 as a two-branched function of y and y_1.

Equating A to zero, we find, for $y \neq 0$,

$$\frac{d}{dx}\,\frac{y_1}{y} = 2\left(\frac{y_1}{y}\right)^{3/2},$$

the solutions of which are given by

(6) $$y = b\,e^{1/(c-x)}$$

and

$$y = b,$$

with b and c constants. The solution $y = 0$, suppressed above, is included among these.

The solutions (6) with $b \neq 0$ are all normal. From the fact that if b stays fixed in (6) at a value distinct from zero, while c approaches ∞ through positive values, y approaches

b uniformly in any bounded domain, we see that the solutions $y = b$ with $b \neq 0$ are seminormal. Consider the solution $y = 0$. Let b have any fixed value distinct from zero. By taking c as a sufficiently small negative number, we can make the second member of (6) and an arbitrarily large number of its derivatives small at pleasure at $x = 0$. This shows that $y = 0$ is a seminormal solution and that the general solution of A is the whole manifold of A.

Of course, by taking b sufficiently small in (6), we can approximate uniformly, with arbitrary closeness, to $y = 0$, by means of normal solutions, in very arbitrary areas. But the discussion above shows what might conceivably happen in other examples.

74. We can extend the preceding results. Let F be any form not in Σ. It can be shown, precisely as in § 13, that if (2) is any solution of Σ, there exists a set of points. dense in $\mathfrak{B}$, such that, given any point a of the set, any positive integer m and any $\epsilon > 0$, there exists a solution $u_1, \cdots, y_p$, analytic at a, for which F does not vanish and for which (3) holds.

It follows that if G is a form with coefficients meromorphic in $\mathfrak{A}$, the coefficients not belonging necessarily to $\mathfrak{F}$, and if G vanishes for every solution of Σ with $F \neq 0$, then G vanishes for every solution of Σ.

Adjunction of new functions to $\mathfrak{F}$

75. Let Σ be a non-trivial closed irreducible system. Assuming $\mathfrak{F}$ not to consist purely of constants, we shall study the circumstances under which Σ can become reducible through the adjunction of new functions to $\mathfrak{F}$, that is, through the replacement of $\mathfrak{F}$ by a field $\mathfrak{F}_1$ of which $\mathfrak{F}$ is a proper subset. The functions of $\mathfrak{F}_1$ are assumed meromorphic in $\mathfrak{A}$.

We form a resolvent for Σ, relative to $\mathfrak{F}$, using a form $Pw - Q$ as in § 28. Let Ω be the system of all forms in the u_i, y_i and w which vanish for all common solutions of Σ and $Pw - Q$ for which $P \neq 0$. Listing the unknowns in the order

$$u_1, \cdots, u_q; \quad w; \quad y_1, \cdots, y_p,$$

we take a basic set

$$(7) \qquad A, A_1, \cdots, A_p$$

for Ω, with A algebraically irreducible in $\mathfrak{F}$. Then $A = 0$ is a resolvent for Σ.

Suppose now that the irreducible factors of A in $\mathfrak{F}_1$ are $B_1, \cdots, B_s$. Then each B_i is of the same order in w as A. For, let r represent the order of A in w. If the coefficients of the powers of w_r in A all had a common factor in $\mathfrak{F}_1$, they would have a common factor in $\mathfrak{F}$, and A would be reducible in $\mathfrak{F}$.

Consider the systems

$$(8) \qquad B_j, A_1, \cdots, A_p$$

$j = 1, \cdots, s$. Let Ω_j be the totality of forms in $\mathfrak{F}_1$ which vanish for every solution of (8) which annuls no separant in (8). Then Ω_j is irreducible in $\mathfrak{F}_1$. Let Σ_j be the system of those forms of Ω_j which are free of w. Then, relative to $\mathfrak{F}_1$, Σ_j is closed and irreducible.

We shall prove that Σ holds every Σ_j, that no Σ_h holds any Σ_k with $k \neq h$, and that every solution of Σ is a solution of some Σ_j. Thus, $\Sigma_1, \cdots, \Sigma_s$ *will be a decomposition of* Σ *in* $\mathfrak{F}_1$, *into essential irreducible systems.*

Since

$$\frac{\partial A}{\partial w_r} = B_2 \cdots B_s \frac{\partial B_1}{\partial w_r} + \cdots + B_1 \cdots B_{s-1} \frac{\partial B_s}{\partial w_r},$$

every normal solution of A is a normal solution of some B_j. Thus every normal solution of (7) is a solution of some Ω_j.

Hence a solution of Σ obtained by suppressing w in a normal solution of (7) is a solution of some Σ_j. Every solution of Σ with $P \neq 0$ is obtained by a suppression of w in some solution of Ω.

Suppose now that some solution of Σ is not a solution of any Σ_j. Let C_j be a form in Σ_j, $j = 1, \cdots, s$, which does not vanish for the solution. Then $C_1 \cdots C_s$ does not vanish for the solution. But $C_1 \cdots C_s$ vanishes for every normal

solution of (7). By § 72, it holds Ω. Hence it vanishes for all solutions of Σ with $P \neq 0$. By § 74, it holds Σ. Thus every solution of Σ is a solution of some Σ_j.

If S, the separant of A, were in some Ω_j, it would be divisible by B_j. Then A and S would have a common factor in $\mathfrak{F}_1$, hence, also, in $\mathfrak{F}$, and A would be algebraically reducible in $\mathfrak{F}$.

Consider any form T of Ω. Any solution of (8), for some j, which annuls neither S nor any separant in (8) is a normal solution of (7) and annuls T. Hence ST is in Ω_j, so that T is in Ω_j. Then every form of Σ is in Σ_j, so that Σ holds Σ_j.

Suppose that P is in some Ω_j. Then the remainder P_1 of P with respect to (7) is in Ω_j. Then P_1 is divisible by B_j, and A and P_1 have a common factor in $\mathfrak{F}$. This is impossible, since P_1 is of lower degree than A in w_r. Then P is not in any Ω_j.

Now $Pw - Q$ is in every Ω_j. It follows easily that Ω_j is the totality of forms with coefficients in $\mathfrak{F}_1$ which vanish for those common solutions of Σ_j and $Pw - Q$ for which $P \neq 0$.

This means that if Σ_h held some Σ_k, where $k \neq h$, then Ω_h would hold Ω_k. Then B_h would be in Ω_k and would be divisible by B_k. Then A would have a double factor in $\mathfrak{F}_1$ and hence would be reducible in $\mathfrak{F}$. Thus no Σ_h can hold a Σ_k with $k \neq h$.

Thus, *for Σ to be reducible relative to $\mathfrak{F}_1$, it is necessary and sufficient that the resolvent of Σ relative to $\mathfrak{F}$ be algebraically reducible in $\mathfrak{F}_1$.*

We see that $B_j = 0$ is a resolvent for Σ_j. Thus in the decomposition of Σ into irreducible systems in $\mathfrak{F}_1$, every essential irreducible system will have $u_1, \cdots, u_q$ as arbitrary unknowns and the sum $r_1 + \cdots + r_p$ of § 31 is the same for all of the irreducible systems.

Indecomposability and irreducibility

76. Let Σ be an indecomposable system of simple forms in $y_1, \cdots, y_n$ the domain of rationality being a field $\mathfrak{F}$. We

shall prove that, *if Σ is considered as a system of differential forms, it is irreducible in $\mathfrak{F}$.*

We assume, as we may, that Σ is non-trivial. Let Λ be the totality of simple forms which hold Σ. Let (1) (with the unknowns relettered) be a basic set for Λ. Let G and H be differential forms such that GH holds Σ. Let G_1 and H_1 be respectively the remainders of G and H with respect to (1). Then $G_1 H_1$ holds Σ.

We shall prove that one of G_1, H_1 is identically zero. Suppose that this is not so. Let G_1 and H_1 be arranged as polynomials in the u_{ij} with $j > 0$, the coefficients being simple forms in $u_1, \cdots, y_p$. We understand that no coefficient is identically zero. The coefficients, being reduced with respect to (1), cannot hold Σ. As Σ is indecomposable, there is a regular solution of (1) which annuls no coefficient. Let a be a value of x for which no coefficient, and no separant or initial, vanishes. For $x = a$, and for the values of $u_1, \cdots, y_p$ in the above solution at a, G_1 and H_1 become polynomials g and h in the u_{ij} with $j > 1$. Let numerical values be assigned to these u_{ij} so that neither g nor h vanishes.

We now construct functions $u_1, \cdots, u_q$, analytic at a, whose values at a are the values in the above solution and whose derivatives appearing in G_1 and H_1 have, at a, the values assigned to the u_{ij} above. For these u_i, (1) has a regular solution in which all u_{ij}, y_i in $G_1 H_1$ have, at a, the values used above. Then $G_1 H_1$ cannot vanish for this regular solution of (1).

Thus, let G_1 vanish identically. Then G vanishes for all regular solutions of (1). But every solution (2) of Σ can be approximated uniformly, in some area, by a regular solution of (1). The derivatives of the functions in (2) which appear in G will be approximated by the corresponding derivatives in the regular solution. Thus G vanishes for (2) and holds Σ. Then Σ is irreducible.

CHAPTER VII

ANALOGUE OF THE HILBERT-NETTO THEOREM THEORETICAL DECOMPOSITION PROCESS

ANALOGUE OF HILBERT-NETTO THEOREM

77. In 1893, Hilbert, extending a result of Netto for polynomials in two variables, proved the following remarkable theorem. *Let* $a_1, \cdots, a_r$; b, *be polynomials in* $y_1, \cdots, y_n$ *with numerical coefficients. Suppose that* b *vanishes for every set of numerical values of* $y_1, \cdots, y_n$ *for which* $a_1, \cdots, a_r$ *all vanish. Then some power of* b *is a linear combination of the* a_i, *with polynomials in* $y_1, \cdots, y_n$ *for coefficients.**

The Hilbert-Netto theorem holds, with no modification of the proof, for simple forms. If $F_1, \cdots, F_r$; G are simple forms in $y_1, \cdots, y_n$ such that G holds the system $F_1, \cdots, F_r$, then some power of G is a linear combination of $F_1, \cdots, F_r$, with simple forms for coefficients.

Assuming the foregoing result, we shall establish the following

THEOREM. *Let* $F_1, \cdots, F_r$; G *be differential forms in* $y_1, \cdots, y_n$, *such that* G *holds the system* $F_1, \cdots, F_r$. *Then some power of* G *is a linear combination of the* F_i *and a certain number of their derivatives, with forms for coefficients.*

78. The above theorem will be easy to prove, with the help of an idea taken from Rabinowitsch's treatment of the algebraic problem, after we have settled a special case.

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* A very simple proof is given by A. Rabinowitsch, Mathematische Annalen, vol. 102 (1929), p. 518. See also van der Waerden, *Moderne Algebra*, vol. 2, p. 11 and Macaulay, *Modular Systems*, p. 48. Hilbert gave results more general than the above.

Suppose that the system $F_1, \cdots, F_r$ has no solutions. We shall show that unity is a linear combination of the F_i and their derivatives, with forms for coefficients.

We assume that unity has no such expression and force a contradiction. First, we shall show that there exist n power series

$$c_{0i} + c_{1i}(x-a) + c_{2i}(x-a)^2 + \cdots, \tag{1}$$

$i = 1, \cdots, n$, which, when substituted for the y_i, render each F_i zero. The series obtained may have zero radii of convergence. The derivatives of the series are thus understood to be obtained formally. In substituing the series for the y_i into a form, we use the Laurent expansions of the coefficients in the form at a. After these formal solutions are secured, we shall be able to show that there exist solutions in which the y_i are actually analytic functions.

79. We consider the system of forms consisting of $F_1, \cdots, F_r$ and of their derivatives of all orders, writing the forms of the infinite system, in any order, in a sequence

$$H_1, H_2, \cdots, H_q, \cdots. \tag{2}$$

Similarly, we write all y_{ij}, arbitrarily ordered, in a sequence

$$z_1, z_2, \cdots, z_q, \cdots. \tag{3}$$

Each H_i will now be considered as a simple form in the z_i. The domain of rationality will be $\mathfrak{F}$.

To show the existence of the formal solutions (1), it will suffice to find a set of numerical values for the z_i, and a value of x at which the coefficients in the F_i are analytic, which make every form in (2) zero.

Consider any non-vacuous finite system Φ of simple forms H_i taken from (2). We shall consider the unknowns in Φ to be those which actually figure in the H_i in Φ.

We know from the Hilbert-Netto theorem, as applied to simple forms, that Φ has solutions. Otherwise unity would be a linear combination of the H_i in Φ, in contradiction of the assumption in § 78.

Let q be any positive integer. We construct, in the following manner, a system Σ_q of simple forms in $z_1, \cdots, z_q$. A simple form K in $z_1, \cdots, z_q$ is to belong to Σ_q if there exists a system Φ, as above, which K holds.* Every Σ_q contains the form 0. When q is so large that an H_i exists involving no z_j with $j > q$, Σ_q will have other forms than 0; for instance, it will contain H_i.

We shall prove that, for every q, Σ_q has solutions. We need consider only the case in which Σ_q has non-zero forms. By § 7, there exists a finite subsystem of Σ_q,

$$K_1, \cdots, K_s \tag{4}$$

which Σ_q holds. With each K_i, there is associated a system Φ_i of forms (2) which K_i holds. The totality of forms in $\Phi_1, \cdots, \Phi_s$ is a finite system Λ of forms (2). Now Λ has solutions and each K_i holds Λ. Hence Σ_q holds Λ, and has solutions. Since every form in $z_1, \cdots, z_q$ which holds Σ_q holds Λ, Σ_q is simply closed.

Evidently, for every q, Σ_q is contained in Σ_{q+1} and consists of those forms in Σ_{q+1} which are free of z_{q+1}.

80. For each q, let Σ_q be decomposed into essential prime systems

$$\Pi_1, \cdots, \Pi_t. \tag{5}$$

Then Σ_q consists of the forms common to all Π_i.†

Let Π' be any prime system in the decomposition (5) of Σ_1. We are going to show that there is a prime system Π'' in the decomposition of Σ_2 whose forms free of z_2 constitute Π'.

Let

$$\Lambda_1, \cdots, \Lambda_v \tag{6}$$

be the decomposition (5) of Σ_2. Those forms of Λ_i which are free of z_2 constitute a prime system Ψ_i. The forms common to

$$\Psi_1, \cdots, \Psi_v \tag{7}$$

* Φ may involve z_i not in K. Of course K need not be in (2).

† It is unnecessary to express, notationally, the dependence of (5) on q.

is the totality of forms in Σ_2 which are free of z_2, that is Σ_1. Then (7) is a decomposition of Σ_1 into prime systems. For, firstly, Σ_1 holds each Ψ_i. Again, if some solution of Σ_1 were not a solution of any Ψ_i, we could find a form S_i in each Ψ_i which does not vanish for the solution. Then $S_1 \cdots S_v$, which is in every Ψ_i, hence in Σ_1, would not vanish for the solution.

Thus the decomposition (5) of Σ_1 is formed from (7) by suppressing certain Ψ_i. Then some Ψ_i is identical with Π'. This means that there is some prime system Π'' in the decomposition (5) of Σ_2 whose forms free of z_2 constitute Π'. Similarly, there is a prime system Π''' in the decomposition (5) of Σ_3 whose forms free of z_3 constitute Π''. We continue, in this way, forming a sequence

$$\Pi', \Pi'', \cdots, \Pi^{(q)}, \cdots. \tag{8}$$

81. We now form a system Ω, putting into Ω every form which is contained in any of the systems (8). Any particular form in Ω involves only a finite number of unknowns.

We are going to find a value a of x for which the coefficients in the F_i are analytic, and numerical values of the z_i, for which every form in Ω with coefficients analytic at a vanishes. Since every H_i of (2) is in Ω, every H_i will vanish for the values found, and we will have the formal solutions (1).

There may be a z_i such that every form in Ω which involves that z_i effectively, also involves some z_j with $j \neq i$. If such z_i exist, we select that one of them whose subscript is a minimum, and call it u_1. It may be that there is some z_i (not u_1) such that no non-zero form of Ω involves only u_1 and the new z_i. If such z_i exist, we represent by u_2 that one of them whose subscript is a minimum.

Continuing in this way, we form a set of unknowns u_i, which is either vacuous, finite or countably infinite, such that no non-zero form in Ω involves only the u_i, while every z_j which is not a u_i appears in a non-zero form involving only that z_j and the u_i.

We order arbitrarily the z_i which are not among the u_i, calling them v_1, v_2, etc. The sequence of v_i, for all that we can say offhand, may be finite or countably infinite. We assume, in what follows, that the v_i are infinite in number; only trivial modifications of language are necessary when their number is finite.

In using the terms "initial", "remainder", etc., below, we shall understand that every u_i precedes every v_j.

From among all non-zero forms in Ω which involve only v_1 and the u_i, we select one, A_1, whose degree in v_1 is a minimum. There exist non-zero forms in Ω which involve only v_2, v_1 and the u_i, and which are reduced with respect to A_1. From among all such forms, we select one, A_2 whose degree in v_2 is a minimum. Continuing, we form an infinite sequence

$$A_1, A_2, \cdots, A_q, \cdots. \tag{9}$$

We are going to show that it is possible to form (9) in such a way that, for every q, the initial I_q of A_q involves only the u_i.

This is true automatically for I_1. Suppose, then, that we have been able to arrange so that $I_1, \cdots, I_{q-1}$ involve only the u_i.

Let B be any non-zero form in Ω, involving $v_1, \cdots, v_q$ and the u_i, reduced with respect to $A_1, \cdots, A_{q-1}$, and of as low a degree in v_q as it can be with these conditions.

The system Φ of all forms in Ω which involve only $v_1, \cdots, v_{q-1}$ and the u_i in $B, A_1, \cdots, A_{q-1}$ is a prime system. This is because Φ is contained in some $\Pi^{(j)}$ and is the system of all forms in that $\Pi^{(j)}$ which involve only the stated u_i, v_i. Then $A_1, \cdots, A_{q-1}$ is a basic set for Φ. We construct a simple resolvent $R = 0$ for Φ, with

$$w = a_1 v_1 + \cdots + a_{q-1} v_{q-1}, \tag{10}$$

the a_i being integers.

The initial Q of B is not in Ω, hence not in Φ. When we replace the v_i in Q by their expressions in terms of w, we get a relation

$$(11) \qquad Q = \frac{P}{S},$$

where P and S involve w und the u_i, the relation holding, where w is as in (10), for every solution of Φ with $S \neq 0$. Then

$$(12) \qquad SQ - P = 0$$

for every solution of Φ, if w is as in (10). As Q is not in Φ, P is not divisible by R. Thus, we have an *identity*

$$(13) \qquad MP + NR = L$$

with L not zero, and free of w, that is, involving only the u_i.

From (12) and (13) we see that, for every solution of Φ, and for w as in (10),

$$(14) \qquad MSQ - L = 0.$$

Then, if w is replaced in (14) by its expression (10), the first member of (14) becomes a form in Φ. We have thus

$$L = UQ + V,$$

with V in Φ and U a form in the unknowns in Φ.

Let B be of degree s in v_q. Let

$$C = UB + Vv_q^s.$$

Then C is in Ω and is of degree s in v_q, with L for initial. The remainder D of C, with respect to $A_1, \cdots, A_{q-1}$, will be of degree s in v_q. Its initial will involve only the u_i. We can use D for A_q in (9). This proves our statement relative to (9) and, in what follows, we assume that every I_q involves only the u_i.

82. We are going to attribute constant values to the u_i in such a way that each I_i becomes a function of x which does not vanish identically.

Each I_i has at most a finite number of factors of the type $u_1 - h$, h constant. Thus the set of polynomials $u_1 - h$

which are factors of one or more I_i is finite or countable. Then let c_1 be a constant such that no I_i is divisible by $u_1 - c_1$. If we put $u_1 = c_1$ in the I_i, each I_i becomes a polynomial J_i, free of u_1 and not identically zero. Similarly we replace u_2 in the J_i by a c_2 so that no J_i vanishes identically. Continuing, we replace all u_i by constants in such a way that each I_i becomes a non-zero function of x.

83. Let $\mathfrak{B}$ be an area in $\mathfrak{A}$ in which the coefficients in the F_i are analytic. Then every H_q in (2) has coefficients analytic in $\mathfrak{B}$. The equation $A_1 = 0$, with the u_i fixed as in § 82, determines one or more functions v_1, of x, analytic in some area $\mathfrak{B}_1$ in $\mathfrak{B}$. Let one of these functions be selected, and substituted into A_2. Then $A_2 = 0$ gives one or more v_2, analytic in $\mathfrak{B}_2$ contained in $\mathfrak{B}_1$. We substitute such a v_2, and the v_1 selected above, into A_3 and solve $A_3 = 0$ for v_3, using an area $\mathfrak{B}_3$ in $\mathfrak{B}_2$. We continue, finding a v_q and a $\mathfrak{B}_q$ for every q. For any q, the functions $v_1, \cdots, v_q$, together with the constant values attributed to the u_i in $A_1, \cdots, A_q$, annul those forms in Ω which involve only the unknowns in $A_1, \cdots, A_q$.

Let a be a point common to all areas $\mathfrak{B}_q$. Then, a, the values of the v_i at a and the constants selected for the u_i, annul those forms in Ω whose coefficients are analytic at a. In particular, the H_i of (2), vanish for the above values.

This proves the existence of the formal solutions (1) of

$$F_1, \cdots, F_r. \tag{15}$$

84. We shall now prove that (15) has analytic solutions. It is not difficult to see that the results of §§ 7–14, and also those of §§ 23, 24 hold when a solution of Σ is defined as any set of series (1) which formally annul every form in Σ.

With this new definition of solution, let (15) be decomposed into closed essential irreducible systems $\Sigma_1, \cdots, \Sigma_s$. We know of course from § 78, that the F_i are not all zero. Let $u_1, \cdots, u_q$ be a set of arbitrary unknowns for Σ_1 and let

$$A_1, \cdots, A_p \tag{16}$$

each A_i introducing y_i, be a basic set for Σ_1. Let S_i be the separant, and I_i the initial, of A_i.

We are going to show that (16) has analytic regular solutions.

Suppose that when the A_i are regarded as simple forms in the y_{ij}, u_{ij} which they involve, they have a solution, consisting of analytic functions, which annuls no S_i or I_i. Then the values of the u_{ij}, y_{ij} at some suitable point will furnish initial conditions for an analytic regular solution of (16).

Now, if

$$T = S_1 \cdots S_p \, I_1 \cdots I_p$$

vanished for all solutions u_{ij}, y_{ij} of (16) considered as a set of simple forms, we would have, using the Hilbert-Netto theorem as applied to simple forms, an identity

$$T^h = C_1 A_1 + \cdots + C_p A_p. \tag{17}$$

But (17) would continue to hold for all formal power series solutions of (16) considered as a set of simple forms. Then the basic set of differential forms (16) would have no regular power series solutions.

This shows that (15) has analytic solutions. We have reached a contradiction which proves that unity is a linear combination of the F_i, and of a certain number of their derivatives, with forms for coefficients.

85. We now complete the proof of the theorem stated in § 77.*

We adjoin an unknown z to the y_i, and consider the system of forms

$$zG - 1, \; F_1, \; \cdots, \; F_r,$$

which evidently has no solutions. Let $K = zG - 1$. Then there exists an identity in the z_j, y_{ij},

$$1 = \sum_{j=0}^{m} C_j \frac{d^j}{dx^j} K + \sum_{j=0}^{m} \sum_{i=1}^{r} D_{ij} \frac{d^j}{dx^j} F_i, \tag{18}$$

where the C_j, D_{ij} are forms in z, y_1, $\cdots$, y_n.

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* Cf. Rabinowitsch, loc. cit.

If we replace z by $1/G$ in (18) and each z_j by the jth derivative of $1/G$, the first sum in (18) vanishes. We find thus an identity

$$1 = \sum_{j=0}^{m} \sum_{i=1}^{r} \frac{E_{ij}}{G^h} \frac{d^j}{dx^j} F_i$$

where the E_{ij} are forms in $y_1, \cdots, y_n$.

Then

$$G^h = \sum_{j=0}^{m} \sum_{i=1}^{r} E_{ij} \frac{d^j}{dx^j} F_i$$

and this establishes our theorem.

Example. We consider two forms in the unknown y,

$$F_1 = y^2, \; F_2 = y_1 - 1.$$

The system F_1, F_2 has no solutions. We have

$$1 = \frac{1}{2} \frac{d^2 F_1}{dx^2} - y \frac{dF_2}{dx} - (y_1 + 1) F_2.$$

86. It follows from § 84 that *if a system of algebraic differential equations in $y_1, \cdots, y_n$ has formal power series solutions, then the system also has analytic solutions.*

One might ask whether a system Σ which is irreducible when the solutions $y_1, \cdots, y_n$ are understood to be analytic functions, remains irreducible when the y_i are allowed to be formal power series. The answer is affirmative. Let GH hold Σ according to the second definition. Then one of G, H holds Σ for the first definition. Suppose that G does. Let

$$(19) \qquad B_1, \cdots, B_s$$

be a finite subsystem of Σ with the same manifold (first definition) as Σ. Then some power of G is a linear combination of the B_i and their derivatives. Then G holds Σ for the second definition.

THEORETICAL PROCESS FOR DECOMPOSING A FINITE SYSTEM OF FORMS INTO IRREDUCIBLE SYSTEMS

87. We deal with any finite system Σ of differential forms in $y_1, \cdots, y_n$. Let p be any positive integer. We denote

by $\Sigma^{(p)}$ the system obtained by adjoining to Σ the first p derivatives of each of its forms.

When the forms in $\Sigma^{(p)}$ are regarded as simple forms in the y_{ij} which they involve, $\Sigma^{(p)}$ goes over into a system $\Lambda^{(p)}$ of simple forms. For domain of rationality, we use $\mathfrak{F}$.

Using the method of §§ 55–60, we decompose $\Lambda^{(p)}$, by a finite number of operations, into essential indecomposable systems

$$(20) \qquad \Phi_1, \cdots, \Phi_r.$$

Let the forms in the Φ_i be considered now as differential forms in the y_i. Then each Φ_i goes over into a system of differential forms Ψ_i. Let any Ψ_i which is held by some Ψ_j with $j \neq i$ be suppressed. This can be accomplished by a finite number of operations (§ 68). There remain systems

$$(21) \qquad \Psi_1, \cdots, \Psi_s.$$

We say that, *for p sufficiently great,* (21) *is a decomposition of Σ into essential irreducible systems.*

88. Let

$$(22) \qquad \Sigma_1, \cdots, \Sigma_t$$

be a decomposition of Σ into finite essential irreducible systems. When the forms in the Σ_i are regarded as simple forms in their y_{ij}, (22) goes over into a system of simple forms

$$(23) \qquad \Gamma_1, \cdots, \Gamma_t.$$

Let us make any selection of t forms, one from each Σ_i, and take their product. Let the products, for all possible selections, be

$$A_1, A_2, \cdots, A_g.$$

Then each A_i holds Σ. By § 77, if p is large, some power of each A_i will be a linear combination of forms in $\Sigma^{(p)}$, with forms for coefficients.

If then each A_i is considered as a simple form in its y_{ij}, and if it is represented then by B_i, each B_i will hold $\Lambda^{(p)}$ if p is sufficiently large. Let p be large enough for this.

We shall prove that each Φ_i of (20) is held by some Γ_j of (23).* Suppose that Φ_1 is not so held. Let C_j be a form of Γ_j, $j = 1, \cdots, t$ which does not hold Φ_1. Then $C_1 \cdots C_t$, that is, some B_i, does not hold Φ_1. Then that B_i cannot hold $\Lambda^{(p)}$. This proves our statement.

It follows that each Ψ_i is held by some Σ_j.

On the other hand, each Σ_i is held by some Ψ_j. Let this be false. Let D_j be a form in Ψ_j, $j = 1, \cdots, r$, (we restore, momentarily, the suppressed Ψ_j) which does not hold Σ_1. Then $G = D_1 \cdots D_r$ does not hold Σ_1. Hence G does not hold Σ. Then, if G is considered as a simple form in its y_{ij}, it does not hold $\Lambda^{(p)}$. This contradicts the fact that $\Lambda^{(p)}$ is equivalent to (20).

Thus, for p sufficiently great, (21) is a decomposition of Σ into essential irreducible systems.

For the above process to become a genuine method of decomposition, it would be necessary to have a method for determining permissible integers p.

This question requires further investigation. In § 89, we treat a special case.

Example 1. Let Σ be $y_1^2 - 4y$, in the unknown y. Then $\Lambda^{(p)}$ is equivalent to the system

$$y_1^2 - 4y,\ y_1(y_2 - 2),\ y_1 y_3 + y_2(y_2 - 2),$$
$$y_1 y_4 + 2y_2 y_3 + y_3(y_2 - 2), \cdots,$$
$$y_1 y_{p+1} + (p-1) y_2 y_p + \cdots + (p-1) y_{p-1} y_3 + y_p(y_2 - 2).$$

$\Lambda^{(1)}$ decomposes into the two indecomposable systems

(24) $$y,\ y_1$$

(25) $$y_1^2 - 4y,\quad y_2 - 2,$$

in the unknowns y, y_1, y_2. If we adjoin $y_1 y_3 + y_2(y_2 - 2)$ to (24), that system decomposes into

(26) $$y,\ y_1,\ y_2$$

(27) $$y,\ y_1,\ y_2 - 2,$$

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* The unknowns are all which appear in (20) and (23).

which are systems in y, y_1, y_2, y_3. The same adjunction to (25) gives (27) and

(28) $$y_1^2-4y, \quad y_2-2, \; y_3.$$

Thus (26), (27) and (28) give the decomposition of $A^{(2)}$. Continuing, we find the decomposition of $A^{(p)}$ to be, for $p>2$,

(29) $$y, \; y_1, \; y_2, \; \cdots, \; y_p,$$

(30) $$y, \; y_1, \; y_2-2, \; y_3, \; \cdots, \; y_p,$$

(31) $$y_1^2-4y, \; y_2-2, \; y_3, \; \cdots, \; y_{p+1}.$$

If we regard the last three systems as systems of differential forms, (31) gives the general solution of y_1^2-4y, while (29) gives the solution $y=0$, which is a second irreducible manifold. The system (30) of differential forms has no solution.

We notice that the system of simple forms y_1^2-4y, y_2-2 holds the system (30) of simple forms. This is in harmony with the fact that every Φ_i in (20) is held be some Γ_j in (23).

Example 2. Let Σ be the form $y_1^2-4y^3$, which, from the fact that its manifold is $y=1/(x-a)^2$ and $y=0$, is seen to be an irreducible system. If we let

$$A_1 = 2y_2-12y^2,$$

and represent the rth derivative of A_1 by A_{r+1}, then $A^{(p)}$ will be

$$y_1^2-4y^3, \; y_1 A_1, \; y_2 A_1+y_1 A_2,$$
$$y_3 A_1+2y_2 A_2+y_1 A_3, \; \cdots,$$
$$y_p A_1+(p-1)y_{p-1}A_2+\cdots+(p-1)y_2 A_{p-1}+y_1 A_p.$$

Then $A^{(1)}$ decomposes into

(32) $$y, \; y_1$$

(33) $$y_1^2-4y^3, \; A_1.$$

We now examine $A^{(2)}$. The adjunction of $y_2 A_1 + y_1 A_2$ to (32) gives the single system

$$(34) \qquad y, \; y_1, \; y_2.$$

The same adjunction to (33) gives

$$(35) \qquad y_1^2 - 4y^3, \quad A_1, \; A_2,$$

and also (34).

Let us examine $A^{(3)}$. The adjunction of $y_3 A_1 + 2y_2 A_2 + y_1 A_3$ to (34) gives the single system

$$(36) \qquad y, \; y_1, \; y_2$$

in the unknowns $y, \cdots, y_4$. The same adjunction to (35) gives

$$(37) \qquad y_1^2 - 4y^3, \; A_1, \; A_2, \; A_3,$$

as well as the system, held by (36), obtained by adjoining y_3 to (36).

Continuing, it is not difficult to prove that the decomposition of $A^{(p)}$ is

$$(38) \qquad y, \; y_1, \; \cdots, \; y_q$$

where q is the greatest integer in $1 + p/2$, and

$$(39) \qquad y_1^2 - 4y^3, \; A_1, \; \cdots, \; A_p.$$

The system (39) of differential forms gives the manifold of Σ, while (38) (differential forms), whose manifold is $y = 0$, is held by (39).

Forms in one unknown, of first order

89. Let A be a form in the single unknown y, of the first order in y, and irreducible algebraically. We shall show how to determine, in a finite number of steps, a finite system of forms whose manifold is the general solution of A.

Let A be of degree m in y_1. We consider the system

$$(40) \qquad A, \; A_1, \; \cdots, \; A_{m-1},$$

there A_j is the jth derivative of A. Let (40) be considered as a system of simple forms, and let it be resolved into finite essential indecomposable systems. There will be precisely one indecomposable system, Δ, which is not held by S, the separant of A (§ 73). Let the forms of Δ be considered now as differential forms in y. Let Σ be the system of differential forms thus obtained.

We shall prove that *the manifold of* Σ *is the general solution of* A.

90. We know that the general solution of A is contained in the manifold of Σ. What we have to show is that every solution of Σ is in the general solution of A.

We observe that A holds Σ. The solutions of A not in the general solution are solutions of S. The common solutions of A and S are solutions of the resultant of A and S with respect to y_1, which is a non-zero form R, of order zero in y. It suffices then to show that every solution u of R which is a solution of Σ is contained in the general solution of A.

Let u_j be the jth derivative of u. Then $A = 0$ for $y = u$, $y_1 = u_1$. There exists an open region $\mathfrak{A}_1$ and an $h > 0$ such that, for

(41) $$x \text{ in } \mathfrak{A}_1 \text{ and } 0 < |y - u| < h,$$

every solution of the algebraic relation $A = 0$, for y_1 considered as a function of y and x, is given by a series

(42) $$y_1 - u_1 = a_0(y-u)^{q/s} + \cdots + a_p(y-u)^{(q+p)/s} + \cdots,$$

where the a_i are functions of x analytic in $\mathfrak{A}_1$ and where q and s are integers, s being positive. The particular series used in the second member of (42) depends on the particular solution y_1 used. But, for each such series, we have $s \leqq m$. We suppose that, in each series, a_0 does not vanish for every x.

The system of functions

(43) $$u, u_1, \cdots, u_m$$

is a solution of Δ. By § 64, there is a region $\mathfrak{A}_2$ in $\mathfrak{A}_1$ in which we can approximate arbitrarily closely to (43) by a

solution of A with R distinct from zero throughout $\mathfrak{A}_2$. We suppose $\mathfrak{A}_2$ to be taken so that the coefficients in A are analytic throughout $\mathfrak{A}_2$.

It follows that, if ξ is any point in $\mathfrak{A}_2$, the differential equation $A=0$ has solutions analytic at ξ, with $R \neq 0$ at ξ, for which $y, \cdots, y_m$ differ arbitrarily slightly at ξ from $u, \cdots, u_m$ respectively.*

Any such solution satisfies (42), in the neighborhood of ξ, for an appropriate choice of the series in (42).† Hence there must be one of the series for which (42) is satisfied by a solution of A with $R \neq 0$ and with $y, \cdots, y_m$ as close as one pleases at ξ to $u, \cdots, u_m$. In what follows, we deal with such a series and assume ξ to taken so that $a_0 \neq 0$ at ξ.

We see first that $q > 0$ in (42). Otherwise $y_1 - u_1$ would not be small at ξ if $y-u$ is small at ξ. Differentiating (42) we find

$$
(44) \qquad \begin{aligned} y_2 - u_2 = & \sum \frac{q+p}{s} a_p (y-u)^{(q+p)/s-1}(y_1-u_1) \\ & + \sum \frac{d a_p}{dx} (y-u)^{(q+p)/s}. \end{aligned}
$$

Replacing $y_1 - u_1$ in (44) by its expression in (42), we find

$$
y_2 - u_2 = \frac{q}{s} a_0^2 (y-u)^{2q/s-1} + b_1 (y-u)^{(2q+1)/s-1} + \cdots,
$$

where the b_i are analytic in $\mathfrak{A}_1$. We notice that, if $m \geq 2$, $2q/s - 1 > 0$. Otherwise $y_2 - u_2$ could not be small at ξ when $y-u$ is small.

Similarly, we find

$$
(45) \qquad \begin{aligned} & y_m - u_m \\ = \frac{q}{s}\left(\frac{2q}{s}-1\right)\cdots\left(\frac{mq}{s}-m+2\right) & a_0^m (y-u)^{mq/s-m+1} + \cdots. \end{aligned}
$$

The coefficient in the first term of the second member is not zero at ξ. Hence the first exponent in the series in (45) must be positive. That is,

---

* Note that if $R \neq 0$ at ξ for a solution of A, then $S \neq 0$ at ξ.

† If $R \neq 0$ at ξ, $y-u \neq 0$ for a neighborhood of ξ.

$$\frac{mq}{s} - m + 1 > 0,$$

so that

$$\frac{mq}{s} - m + 1 \geqq \frac{1}{s}.$$

Thus,

$$q \geqq s - \frac{s}{m} + \frac{1}{m},$$

and, as $s \leqq m$, we have $q > s - 1$, so that $q \geqq s$.

We are now able to show that u belongs to the general solution of A. In (42), we replace $y - u$ by v^s. Then (42) goes over into the differential equation

$$(46) \qquad s\frac{dv}{dx} = a_0 v^{q-s+1} + \cdots + a_p v^{q+p-s+1} + \cdots.$$

Since the second member of (46) is analytic in v and x for v small and x close to ξ, then, if we fix v as a small quantity at ξ, distinct from 0, (46) will have a solution analytic at ξ, not identically zero, and with any desired finite number of derivatives as small as one pleases at ξ.* Then $y - u = v^s$, while not zero at ξ, will be small at ξ, together with as great a finite number of its derivatives as one may choose to consider. Solutions of A, close to u, but distinct from u, at ξ, cannot make $R = 0$.

Thus, if u is a solution of S, as well as of R, u is a seminormal solution of A and belongs to the general solution of A. If u is not a solution of S, u certainly belongs to the general solution.

* Equation (46) is satisfied by $v = 0$, and its solution is analytic in the constant of integration.

CHAPTER VIII

ANALOGUE FOR FORM QUOTIENTS OF LÜROTH'S THEOREM

91. It is an important theorem of Lüroth that if α and β are rational functions of x, then α and β are rational functions of a third rational function, γ, which, in turn, is a rational combination of α and β.*

We are going to prove the following analogue of Lüroth's theorem.

THEOREM. *Let α and β be two form quotients* (§ 38) *in a single unknown y. Then there exists a form quotient γ, in y, such that*

(a) *α and β are rational combinations of γ and of a certain number of its derivatives,*

(b) *γ is a rational combination of α, β and a certain number of their derivatives.*

The coefficients in the rational combinations are functions of x in $\mathfrak{F}$.

As to the degree of uniqueness of γ, we prove that if γ_1 and γ_2 are two possibilities for γ, then $\gamma_2 = (a\gamma_1 + b)/(c\gamma_1 + d)$, with a, b, c, d functions of x in $\mathfrak{F}$.

92. We prove the following lemma.

LEMMA. *Let $P_1, \cdots, P_m, Q, R$ be forms in y, R not identically zero. Suppose that the relations*

$$\frac{P_i(y)}{R(y)} = \frac{P_i(z)}{R(z)}, \qquad i = 1, \cdots, m, \tag{1}$$

where y and z are analytic functions for neither of which R vanishes, imply the relation

---

* Appel et Goursat, *Fonctions Algébriques*, 2nd edition, vol. 1, p. 283. Van der Waerden, *Moderne Algebra*, vol. 1, p. 126.

$$(2) \qquad \frac{Q(y)}{R(y)} = \frac{Q(z)}{R(z)}.$$

Then the form quotient Q/R is a rational combination of the P_i/R, and of a certain number of their derivatives, with coefficients in $\mathfrak{F}$.

The values 1 and 2 of n_i will suffice in our applications of this lemma.

Let $v_1, \cdots, v_m$; w be new unknowns. Consider the forms

$$(3) \qquad Rv_i - P_i \ (i = 1, \cdots, m); \qquad Rw - Q.$$

As in § 32, the system Σ of all forms in the v_i, w, y which vanish for all solutions of (3) with $R \neq 0$, is irreducible. It is easy to prove that any one unknown in Σ is a set of arbitrary unknowns.

We take v_1 as arbitrary unknown, and form a basic set for Σ

$$(4) \qquad A_2, \cdots, A_m, B, C$$

which introduces, in succession, $v_2, \cdots, v_m, w, y$.

We are going to prove that B is of order zero in w, and, indeed, that it is linear in w.

Suppose that B is of order greater than zero in w. Consider any regular solution of (4) with $R(y) \neq 0$. Such solutions exist because $R(y)$ is not in Σ. Without disturbing the v_i in the solution, we can alter the initial conditions for w slightly at some point and get a second regular solution with $R(y) \neq 0$. This would be contrary to the hypothesis of the lemma.

We shall prove now that B is linear in w. Suppose that this is not so.

Let $A_2, \cdots, A_m$ be of the respective orders $r_2, \cdots, r_m$ in $v_2, \cdots, v_m$. Let C be of order r in y.

In (4), we replace the symbols

$$v_{2r_2}, \cdots, v_{mr_m}, w, y_r$$

by

$$z_2, \cdots, z_m, z_{m+1}, z_{m+2}$$

respectively. The remaining v_{ij} and y_i we replace, in any order, by symbols u_i (see § 65).

Then (4) goes over into a basic set

$$(5) \qquad F_2, \cdots, F_m, D, E$$

of a prime system.

Let

$$(6) \qquad \zeta'', \cdots, \zeta^{(m)}$$

be analytic functions of the u_i and x which annul $F_2, \cdots, F_m$ when substituted for $z_2, \cdots, z_m$ (§ 45). As D is of degree at least 2 in z_{m+1}, we can get two distinct functions, $\zeta_1^{(m+1)}$ and $\zeta_2^{(m+1)}$ which, with (6), annul D (§ 46). After treating E we will have two sets of functions

$$(7) \qquad \begin{array}{l} \zeta'', \cdots, \zeta^{(m)}, \zeta_1^{(m+1)}, \zeta_1^{(m+2)}, \\ \zeta'', \cdots, \zeta^{(m)}, \zeta_2^{(m+1)}, \zeta_2^{(m+2)} \end{array}$$

which annul (5) when substituted for the z_i. No separant or initial in (5) is anulled by either set (7).

Let T be the remainder of R with respect to (4). Let new u_i be taken, to correspond to the v_{1i} in T which are not in (4). Let T go over into a simple form U in the u_{ij}, z_i. As U is reduced with respect to (5), it will not be annulled by either set (7).

We attribute numerical values to x and the u_i in the following way. We require the functions (7) and the coefficients in (5) and in U to be analytic for these values. We require, secondly, that $\zeta_1^{(m+1)} - \zeta_2^{(m+1)} \neq 0$. Finally, we ask that U and the separants in (5) do not vanish for either set (7), for these values.

The values chosen furnish initial conditions for two solutions of (4) which annul neither R nor any separant. We use the same v_1 in both solutions. Then $v_2, \cdots, v_m$ are the same for both solutions. On the other hand, w will not be the same in both solutions. This contradicts the hypothesis of the lemma.

Then B is linear in w. If we replace v_i in B by P_i/R and w by Q/R, the resulting expression in y must vanish identically. This completes the proof of the lemma.

93. Let $A(y)$ and $B(y)$ be two non-zero forms in y, relatively prime as polynomials in the y_i and not both free of y. Let r be the maximum of their orders in y. We shall prove that

$$A(y)\,B(z) - B(y)\,A(z) \tag{8}$$

is not divisible by any form, not a function of x, which does not effectively involve y_r.

Suppose that there is such a factor, C, free of y_r. Let

$$A(y) = M_0\,y_r^h + \cdots + M_h, \quad B(y) = N_0\,y_r^h + \cdots + N_h,$$

where it is possible that either M_0 or N_0 is zero. Then C must be a factor, for every i, of

$$M_i\,B(z) - N_i\,A(z).$$

We shall show that C cannot be a form in z alone. Suppose that C involves only z. Since $A(y)/B(y)$ is not a function of x, we can assign two distinct sets of rational numerical values to $y, \cdots, y_r$ in such a way that

$$A_1\,B_2 - B_1\,A_2,$$

where the subscripts correspond to the substitutions, is not zero. As, by (8)

$$A_1\,B(z) - B_1\,A(z), \qquad A_2\,B(z) - B_2\,A(z)$$

are divisible by C, then $B(z)$ and $A(z)$ must both be divisible by C. Thus C would have to be a function of x.

Let C be of order $s \geqq 0$ in y. Let $\alpha = A(z)/B(z)$ and let

$$w_i = M_i - \alpha\,N_i, \qquad i = 0, \cdots, h.$$

For any rational numerical values of $z, \cdots, z_r$ for which $B(z) \neq 0$, and for which the coefficient of the highest power

of y_s in C does not vanish, the expressions w_i will all have a common factor, which will be a polynomial in the y_i, with coefficients in $\mathfrak{F}$.

Let

$$w_u = u_0 w_0 + \cdots + u_h w_h; \qquad w_v = v_0 w_0 + \cdots + v_h w_h,$$

where the u_i, v_i are indeterminates. Then, for arbitrary rational u_i and v_i, and for rational z_i as above, w_u und w_v will both be divisible by a polynomial effectively involving y_s.

Then the resultant ϱ of w_u and w_v with respect to y_s must vanish identically in the u_i, v_i and x, y, $\cdots$, y_{r-1} with y_s omitted, if α is obtained by the indicated substitutions for the z_i.

Now ϱ is a polynomial in α. Since α depends effectively on the z_i, we can find an infinite system of sets of numerical values for the z_i, as described above, each set giving a distinct result for α. Thus ϱ is identically zero, even in α.

Then w_u and w_v have a common factor which is a polynomial in the u_i, v_i, y_i and α, with coefficients in $\mathfrak{F}$. Thus, the expressions w_i, with α indeterminate, must all have a common factor δ, which is a polynomial in y, $\cdots$, y_{r-1}, α with coefficients in $\mathfrak{F}$.

If δ were free of α, every M_i and every N_i would be divisible by δ. Then A and B would not be relatively prime. Thus δ is of the first degree in α.

Then, for every i, we have

$$M_i B(z) - N_i A(z) = R_i[EB(z) + FA(z)],$$

with R_i, E, F forms in y. Then (8) has a factor

$$R_0 y_r^h + \cdots + R_h,$$

which is free of z. This is impossible, for the same reason for which C, above, could not be a form in z alone. The proof is completed.

94. We proceed with the proof of the theorem stated in § 91. When α and β are both free of y, we take $\gamma = 1$. In what follows, we assume that α and β are not both free of y. We may write α and β with a common denominator. Let $\alpha = P/R$, $\beta = Q/R$. Consider the forms in y and z

$$\begin{aligned}&P(y)\,R(z) - P(z)\,R(y),\\ &Q(y)\,R(z) - Q(z)\,R(y)\end{aligned} \tag{9}$$

which are not both identically zero.

Let $\Sigma_1, \cdots, \Sigma_s$ be a decomposition of (9) into closed essential irreducible systems. From each Σ_i, we select a non-zero G_i which is of as low a rank as possible in z. We assume, as we may, that each G_i is algebraically irreducible. There must be some G_i which involves both y and z effectively. Otherwise, (9) would imply a relation of the type

$$C(y)\,D(z) = 0.$$

For $z = y$, this would become $C(y)\,D(y) = 0$. But (9) is satisfied for $z = y$ with y arbitrary.

Let $G_1, \cdots, G_p$ involve both y and z, while $G_{p+1}, \cdots, G_s$ involve either y alone or z alone. For $i \leqq p$, the manifold of Σ_i is the general solution of G_i.

Let $G_{p+1}, \cdots, G_q$ involve only z, and $G_{q+1}, \cdots, G_s$ involve only y. Let

$$M = G_{p+1} \cdots G_q.$$

Then M is not in any Σ_i, $i \leqq p$, because, in the general solution of G_i, $i \leqq p$, z can be taken almost arbitrarily. That is, if we take the unknowns in G_i in the order z, y, then, given any regular solution of G_i, we can modify z and any number of its derivatives, at some point, slightly, but otherwise arbitrarily, and get a second regular solution of G_i.

Let the greatest of the orders of $G_1, \cdots, G_p$ in z be r. Let G_i be of order r in z for $i \leqq m$, and of order less than r for $m < i \leqq p$.

Let

$$H = G_1 \cdots G_m. \tag{10}$$

We write H as a polynomial in z_r. Let

$$H = F z_r^h + F_1 z_r^{h-1} + \cdots + F_h. \tag{11}$$

There must be some ratio F_i/F which is not independent of y. Otherwise some factor of F would be a factor of

every F_i and H would have an irreducible factor not involving z_r. This would contradict (10). Let F_t/F be not independent of y.

Let $K = \partial H/\partial z_r$. Because $G_1, \cdots, G_m$ are algebraically irreducible forms, and none of them divisible by any other, the resultant U of H and K with respect to z_r is not identically zero.

Understanding that the unknowns in H have the order y, z, let B be the remainder of M with respect to H. We say that B and H are relatively prime polynomials. If, for instance, B were divisible by G_1, some $K^g F^h M$ would hold Σ_1. Now F is not in Σ_1, since it is of lower rank than G_1 in z. Again,

$$(12) \quad K = \frac{\partial G_1}{\partial z_r} G_2 \cdots G_m + \cdots + \frac{\partial G_m}{\partial z_r} G_1 \cdots G_{m-1}.$$

Each term after the first in the second member of (12) is in Σ_1. The first term is not. Thus K is not in Σ_1. As M is not in Σ_1, H and B are relatively prime.

Thus the resultant V of H and B with respect to z_r is not identically zero.

Consider the form

$$S = UVFG_{m+1} \cdots G_p.$$

We can assign rational numerical values to $z, \cdots, z_{r-1}$ so that S becomes a non-zero form T in y, and so that F_t/F becomes a form quotient γ in y which is not a function of x.

We say that γ as thus determined satisfies the conditions of § 91.

95. Let

$$(13) \qquad L(y) = TR(y)\,G_{q+1} \cdots G_s.$$

Let $\bar{y}$ and $\tilde{y}$ be two functions of x, analytic in some part of $\mathfrak{A}$, for neither of which L vanishes, and for which

$$\frac{P(\bar{y})}{R(\bar{y})} = \frac{P(\tilde{y})}{R(\tilde{y})}; \quad \frac{Q(\bar{y})}{R(\bar{y})} = \frac{Q(\tilde{y})}{R(\tilde{y})}.$$

Functions $\bar{y}$, $\tilde{y}$ exist. For instance, we can use any $\bar{y}$ such that $L(\bar{y}) \neq 0$ and then take $\tilde{y} = \bar{y}$. Let a be any value of x for which $\bar{y}$, $\tilde{y}$ and the coefficients in R and every G_i, $i = 1, \cdots, s$ are analytic, and for which $L(\bar{y})\,L(\tilde{y}) \neq 0$.

By the definition of $L(y)$, the equation $H = 0$, where $z, \cdots, z_{r-1}$ are replaced by the rational values used above, x by a and y and its derivatives by the corresponding values for $\bar{y}$ at a, will determine h distinct values of z_r.* To each such z_r will correspond a regular solution,† with $y = \bar{y}$, of some G_i, $i \leqq m$, which is not a solution of any G_i with $i > m$. (See (12)).

A similar result holds for $\tilde{y}$.

Now, for y equal to $\bar{y}$ or to $\tilde{y}$, (9) will have the same solutions in z. Let $z, \cdots, z_{r-1}$ have, at a, the rational values used above. With these initial conditions for z, no solution $(\bar{y}, z)$ or $(\tilde{y}, z)$ of (9) will be a solution of a Σ_i with $m < i \leqq p$ or $q < i \leqq s$. There will be h solutions $(\bar{y}, z)$ and h solutions $(\tilde{y}, z)$, which annul the form H. None of these $2h$ solutions will annul $M(z)$ and they will be the only solutions of (9) with $y = \bar{y}$ or $\tilde{y}$ and with $z, \cdots, z_{r-1}$ as indicated at a, for which $M(z) \neq 0$. Thus, in the h solutions $(\bar{y}, z)$ which annul H, the h functions z must be the same as in the $(\tilde{y}, z)$ which annul H.

Then, for $y = \bar{y}$ or for $y = \tilde{y}$, the numerical equation $H = 0$ for z_r must have the same h roots for z_r. We may let the value a, of x, range over an area. Thus, in the expression for H in (11), every ratio F_i/F, with $z, \cdots, z_{r-1}$ rational as above, must be, for every x, the same for $\bar{y}$ as for $\tilde{y}$.

Consider then γ. For

$$\alpha(\bar{y}) = \frac{P(\bar{y})\,L(\bar{y})}{R(\bar{y})\,L(\bar{y})} = \frac{P(\tilde{y})\,L(\tilde{y})}{R(\tilde{y})\,L(\tilde{y})} = \alpha(\tilde{y})$$

and for $\beta(\bar{y}) = \beta(\tilde{y})$, similarly, we have

$$\gamma(\bar{y}) = \gamma(\tilde{y}).$$

* Consider that S is divisible by FU.

† For the order y, z.

By § 92, γ is a rational combination of α, β and a certain number of their derivatives with coefficients in $\mathfrak{F}$.

96. We prove now that α and β are rational in γ and its derivatives.

In F_t and F, with $z, \cdots, z_{r-1}$ rational as above, let y be replaced by z. There will result two forms, $A_t(z)$ and $A(z)$. We wish to show that neither A_t nor A has a higher rank in z than H.

Evidently, it is enough to show that if I is the form obtained from H by interchanging y and z, then I is not of higher rank than H in z.

For $i \leqq p$, let E_i be the form which results from G_i when y and z are interchanged. Then $E_1, \cdots, E_p$ must be multiples of $G_1, \cdots, G_p$ taken in some order. For, since (9) is symmetrical in y and z, the interchange of y and z in $\Sigma_1, \cdots, \Sigma_s$ will accomplish an interchange in pairs of those systems. No Σ_i with $i \leqq p$ can be converted into a Σ_j with $j > p$. Otherwise Σ_i would contain a form in y alone or in z alone. This is impossible, for as was seen above, either of y and z can be taken almost arbitrarily in the general solution of G_i with $i \leqq p$.

Then $\Sigma_1, \cdots, \Sigma_p$ are permuted among themselves. If Σ_i and Σ_j are interchanged, then E_i must hold Σ_j. Then E_i is not of lower order than G_j either in z or in y. By symmetry, E_i and G_j have the same order in z. Then E_i must be divisible by G_j, so that, as E_i and G_j are algebraically irreducible, their ratio is a function of x in $\mathfrak{F}$.

As G_i has a greater rank in z than G_j if $i \leqq m$ and $j > m$, it follows that I is not of higher rank in z than H.

Let C be a highest common factor for A_t and A. Let $A_t = CW_t$ and $A = CW$ with W_t and W relatively prime. We are going to show that

$$J = W_t(y)\,W(z) - W(y)\,W_t(z),$$

which is not zero, equals H multiplied by a function of x.

We observe first that J is not of higher rank than H in z.

Taking the unknowns in H in the order y, z, we let Y be the remainder of $L(z)$, (see (13)), with respect to H.

Then Y and H are relative prime. Let Z be the resultant of H and Y with respect to z_r. Then Z is not zero.

Let real or complex numerical values be assigned to $z, \cdots, z_{r-1}$ in such a way that UZF does not vanish identically in y and its derivatives. Let

$$L_1(y) = L(y)\, UZF,$$

where the substitutions just indicated have been made in U, Z, F. We observe that the coefficients in L_1 may not be in $\mathfrak{F}$.

Then, if y is an analytic function for which $L_1(y) \neq 0$, and if a is a suitably chosen value of x, for which $L_1(y) \neq 0$, the differential equation $H = 0$, with $z, \cdots, z_{r-1}$ as just taken, at a, will determine h distinct functions z for which $L(z) \neq 0$ and for each of which one has

$$\alpha(z) = \alpha(y); \qquad \beta(z) = \beta(y).$$

Hence, for the chosen y and for each such z, one has $J = 0$.

If we modify the numerical values attributed to $z, \cdots, z_{r-1}$ slightly, but arbitrarily, and make arbitrarily slight variations in the values of $y, \cdots, y_r$ at a, leaving the higher derivatives of y alone, we will still have $L_1(y) \neq 0$ and we will get h new functions z, which, with the new y, will make J zero.

All in all, we see that in some open region in the space of $x, y, \cdots, y_r; z, \cdots, z_{r-1}$, the equation $J = 0$ for z_r admits all of the roots of the equation $H = 0$ for z_r. Then, as H has no repeated factors, J is divisible by H. As J is not of higher rank than H in z and as J has no factors (not functions in $\mathfrak{F}$) which do not involve z_r (§ 93), we have $J = \mu H$, with μ a function of x in $\mathfrak{F}$.

Let N be the remainder for $R(z)$ with respect to H for the order y, z. The resultant X of H and N with respect to z_r is not identically zero. Let rational values be substituted for $z, \cdots, z_{r-1}$ in X and in FU of § 94 so that XFU does not vanish. Let D be the non-zero form in y which $R(y)\, XFU$ becomes for these substitutions.

Now, let $\bar{y}$ and $\tilde{y}$ be functions for which

$$(14) \qquad W(\bar{y})\, D(\bar{y})\, W(\tilde{y})\, D(\tilde{y}) \neq 0,$$

and which, when substituted for y and z in J, render J zero. Let a be a value of x for which (14) does not vanish. The differential equation $H = 0$ for z, with $y = \bar{y}$, has h solutions z with $z, \cdots, z_{r-1}$ assuming the above rational values at a, and with $R(z) \neq 0$. The h pairs of functions $(\bar{y}, z)$ thus obtained are solutions of (9). Similarly, we get h pairs $(\tilde{y}, z)$ which are solutions of H and of (9). It is easy to see, because $H = J/\mu$, that the functions z in the h pairs $(\bar{y}, z)$ are the same as those in the $(\tilde{y}, z)$. It follows that

$$\alpha(\bar{y}) = \alpha(\tilde{y}), \quad \beta(\bar{y}) = \beta(\tilde{y}).$$

Then α and β are rational combinations of $W_t(y)/W(y)$ and its derivatives. As $\gamma = W_t(y)/W(y)$, the theorem of § 91 is proved.

The above proof, and the methods of Chapter V, contain everything essential for the construction of γ in a finite number of steps.

97. Suppose that we have two form quotients like γ above, γ_1 and γ_2. We see immediately that γ_2 is rational in terms of γ_1 and its derivatives and that γ_1 is similarly expressible in γ_2.

We now apply the method of § 92. Let $v_1 = \gamma_1$, $v_2 = \gamma_2$. We see that the system Σ of all forms in v_1, v_2 which vanish identically in y for $v_1 = \gamma_1$, $v_2 = \gamma_2$ is irreducible. Let A be a non-zero form of Σ, of a minimum rank in v_2. According to § 92 (γ_1 can be made to correspond to P_1/R and γ_2 to Q/R), A is of zero order in v_2, and is linear in v_2. Similarly if B is a non-zero form of Σ of a minimum rank in v_1, then B is linear in v_1. If we take A and B algebraically irreducible, as we may, each will be divisible by the other. Hence A is linear both in v_1 and in v_2. This proves that γ_1 and γ_2 are linear fractional combinations of each other.

CHAPTER IX

RIQUIER'S EXISTENCE THEOREM FOR ORTHONOMIC SYSTEMS

98. In Chapter X, we shall extend some of the main results of the preceding chapters to systems of algebraic partial differential equations. We shall find it necessary to use an important existence theorem due to Riquier. We develop this existence theorem now, following, in some respects, the concise exposition of Riquier's work given by J. M. Thomas.*

For the proof, in Chapter X, that every system is equivalent to a finite number of irreducible systems, only § 106 of the present chapter, which can be read immediately, is necessary.

MONOMIALS

99. We deal with m independent variables, $x_1, \cdots, x_m$. By a *monomial*, is meant an expression $x_1^{i_1} \cdots x_m^{i_m}$, where the i_k are non-negative integers. If $\alpha = \gamma\beta$, with α, β, γ monomials, then α is called a *multiple* of β. Given two distinct monomials,

$$x_1^{i_1} \cdots x_m^{i_m}, \qquad x_1^{j_1} \cdots x_m^{j_m},$$

the first is said to be *higher* or *lower* than the second according as the first non-zero difference $i_k - j_k$ is positive or is negative.

The following theorem, due to Riquier, is used only in Chapter X.

THEOREM: *Let*

$$(1) \qquad \alpha_1, \alpha_2, \cdots, \alpha_q, \cdots$$

be an infinite sequence of monomials. Then there is an α_i *which is a multiple of some* α_j *with* $j < i$.

* Annals of Mathematics, vol. 30 (1929), p. 285. References will be found in this paper.

Let β_1 be one of those α_i for which the exponent of x_1 is a minimum. Consider the monomials which come after β_1 in (1). Let β_2 be a monomial of this class whose degree in x_1 does not exceed that of any other monomial of the class. Of the monomials which follow β_2, let β_3 be one of minimum degree in x_1. Continuing, we form an infinite sequence of monomials

$$\beta_1, \beta_2, \beta_3, \cdots \tag{2}$$

whose degrees in x_1 are non-decreasing. We extract similarly, from (2), a sequence in which the degrees in x_2 do not decrease. We arrive finally at an infinite subsequence of (1) in which each monomial is a multiple of all which precede it.

Dissection of a Taylor series

100. Let

$$\sum \frac{a_{i_1 \cdots i_m}}{i_1! \cdots i_m!} x_1^{i_1} \cdots x_m^{i_m} \tag{3}$$

be the Taylor expansion at

$$x_i = 0, \qquad i = 1, \cdots, m \tag{4}$$

of a function u of $x_1, \cdots, x_m$ analytic at the point (4). Let $[\alpha]$ be any given finite and non-vacuous set of distinct monomials. We are going to separate (3), with respect to $[\alpha]$, into a set of components.

Let a be the greatest exponent of x_1 in the set $[\alpha]$. We write

$$u = f_0 + x_1 f_1 + x_1^2 f_2 + \cdots + x_1^{a-1} f_{a-1} + x_1^a f_a, \tag{5}$$

where, for $i < a$, $x_1^i f_i$ contains all terms in (3) in which the exponent of x_1 is precisely i. As to $x_1^a f_a$, it contains all terms divisible by x_1^a. Then $f_1, \cdots, f_{a-1}$ are series in $x_2, \cdots, x_m$, while f_a involves also x_1.*

We define sets of monomials $[\alpha]_\lambda$, $\lambda = 0, \cdots, a$, as follows. If $[\alpha]$ contains monomials in which the exponent of x_1 does

* We consider every combination $i_1, \cdots, i_m$ to occur in (3), using zero coefficients if necessary.

not exceed λ, then $[\alpha]_\lambda$ is to consist of all such monomials in $[\alpha]$. If there are no such monomials, then $[\alpha]_\lambda$ is to be unity. Let $[\beta]_\lambda$ be the set of monomials in $x_2, \cdots, x_m$ obtained by putting $x_1 = 1$ in $[\alpha]_\lambda$. We now give to each f_λ, with respect to x_2, the treatment accorded to u, above, with respect to x_1. For $\lambda < a$, we get a representation of the type

$$f_\lambda = f_{\lambda 0} + x_2 f_{\lambda 1} + \cdots + x_2^b f_{\lambda b}, \tag{6}$$

where b depends upon λ, the $f_{\lambda i}$ with $i < b$ involving $x_3, \cdots, x_m$, while $f_{\lambda b}$ involves also x_2. For $\lambda = a$, each f_{ai} involves x_1. That is, in the dissection of f_a, we treat x_1 like $x_3, \cdots, x_m$.

We now operate on each $f_{\lambda\mu}$ with respect to x_3. We use a set of monomials $[\gamma]_{\lambda\mu}$, where, if $[\beta]_\lambda$ has monomials of degree not exceeding μ in x_2, $[\gamma]_{\lambda\mu}$ is obtained by putting $x_2 = 1$ in all such monomials, and where, otherwise, $[\gamma]_{\lambda\mu}$ is unity.

Continuing, we find an expression for u,

$$u = \sum x_1^{i_1} \cdots x_m^{i_m} f_{i_1 \cdots i_m}, \tag{7}$$

the summation extending over a finite number of terms.

Example: Let u be a function of x, y, z. Let $[\alpha]$ be

$$xz^2, \quad xy, \quad x^2yz.$$

For x, we find

$$u = f_0(y, z) + x f_1(y, z) + x^2 f_2(x, y, z).$$

We now treat each f_i with respect to y, the set of monomials being that indicated below;

$$\begin{array}{ll} f_0(y, z) & 1; \\ f_1(y, z) & z^2, \ y; \\ f_2(x, y, z) & z^2, \ y, \ yz. \end{array}$$

Hence

$$\begin{aligned} f_0(y, z) &= f_{00}(y, z), \\ f_1(y, z) &= f_{10}(z) + y f_{11}(y, z), \\ f_2(x, y, z) &= f_{20}(x, z) + y f_{21}(x, y, z). \end{aligned}$$

The final step is

$$\begin{array}{lll}
f_{00}(y, z) & = f_{000}(y, z) & 1; \\
f_{10}(z) & = f_{100} + zf_{101} + z^2 f_{102}(z) & z^2; \\
f_{11}(y, z) & = f_{110}(y) + zf_{111}(y) + z^2 f_{112}(y, z) & 1, z^2; \\
f_{20}(x, z) & = f_{200}(x) + zf_{201}(x) + z^2 f_{202}(x, z) & z^2; \\
f_{21}(x, y, z) & = f_{210}(x, y) + zf_{211}(x, y) + z^2 f_{212}(x, y, z) & 1, z, z^2.
\end{array}$$

Thus the dissection of u is

$$\begin{aligned}
u = {} & f_{000}(y, z) + xf_{100} + xzf_{101} + xz^2 f_{102}(z) \\
& + xyf_{110}(y) + xyzf_{111}(y) + xyz^2 f_{112}(y, z) \\
& + x^2 f_{200}(x) + x^2 zf_{201}(x) + x^2 z^2 f_{202}(x, z) \\
& + x^2 yf_{210}(x, y) + x^2 yzf_{211}(x, y) + x^2 yz^2 f_{212}(x, y, z).
\end{aligned}$$

101. Consider any monomial $\alpha = x_1^{j_1} \cdots x_m^{j_m}$ in $[\alpha]$ and any monomial β in the expansion of u which is a multiple of α. Of course, β appears in one and in only one of the terms in the second member of (7). Let it appear in $x_1^{i_1} \cdots x_m^{i_m} f_{i_1 \cdots i_m}$. We shall prove that $x_1^{i_1} \cdots x_m^{i_m}$ is a multiple of α. For $m = 1$, this result certainly holds. Let the result be true for $m = r - 1$. We shall prove it for $m = r$. We observe first that in the resolution (5) of u, β appears in a term $x_1^{i_1} f_{i_1}$ with $i_1 \geqq j_1$.

Suppose first that $i_1 < a$ in (5). Then $\beta / x_1^{i_1}$ is free of x_1. Among the monomials used in the dissection of f_{i_1} will be $x_2^{j_2} \cdots x_r^{j_r}$ and $\beta / x_1^{i_1}$ will be a multiple of $x_2^{j_2} \cdots x_r^{j_r}$. As there are only $r - 1$ variables involved now, $\beta / x_1^{i_1}$ will appear in a term $\epsilon f_{i_1 i_2 \cdots i_r}$ in the dissection (7) of f_{i_1} with ϵ divisible by $x_2^{j_2} \cdots x_r^{j_r}$. Thus $x_1^{i_1} \cdots x_r^{i_r}$ is divisible by α.

Suppose now that $i_1 = a$. Then β / x_1^a is contained in f_a. Among the monomials used in the dissection of f_a will be $x_2^{j_2} \cdots x_r^{j_r}$. Now the formal scheme in (7) of the dissection of f_a can be obtained by taking a function g of $x_2, \cdots, x_r$, dissecting g with respect to the monomials associated with f_a and then adjoining x_1 to the variables in the series yielded by g. That is, the monomials $x_2^{i_2} \cdots x_r^{i_r}$ in the dissections,

analogous to (7), of f_a and g, will be the same. Let γ result from β on putting $x_1 = 1$. Then γ is found in the dissection of g with an $x_2^{i_2} \cdots x_r^{i_r}$ divisible by $x_2^{j_2} \cdots x_r^{j_r}$. The same would therefore be true for β/x^a in the dissection of f_a. This completes the proof.

It follows that every monomial in $[\alpha]$ is an $x_1^{i_1} \cdots x_m^{i_m}$ in (7).

102. The set of monomials consisting of all $x_1^{i_1} \cdots x_m^{i_m}$ in (7) which are multiples of monomials in $[\alpha]$ will be called the *extended set arising from* $[\alpha]$. The set of monomials $x_1^{i_1} \cdots x_m^{i_m}$ in (7) not in the extended set will be called the set *complementary* to $[\alpha]$.

If $[\alpha]$ is identical with the extended set arising from $[\alpha]$, then $[\alpha]$ will be called *complete.*

Consider a set $[\alpha]$ which is not complete. We shall prove that it is possible to form a complete set by adjoining to $[\alpha]$ multiples of monomials in $[\alpha]$.

Let p be the maximum of all exponents in all monomials in $[\alpha]$. Then, in (7), no i_k exceeds p.

Let $[\alpha]'$ be the extended set arising from $[\alpha]$. Then if $[\alpha]'$ is not complete, it is a proper subset of its extended set $[\alpha]''$ (§ 101). Since we can never get more than $(p+1)^m$ monomials $x_1^{i_1} \cdots x_m^{i_m}$ in (7), this process of taking extended sets must bring us eventually to a complete set.

103. In (7), the variables in an $f_{i_1 \cdots i_m}$ will be called the *multipliers* of the corresponding $x_1^{i_1} \cdots x_m^{i_m}$, and all other variables will be called *non-multipliers* of $x_1^{i_1} \cdots x_m^{i_m}$. Of course, if $f_{i_1 \cdots i_m}$ is a constant, $x_1^{i_1} \cdots x_m^{i_m}$ has no multipliers.

Let $\beta = x_1^{i_1} \cdots x_m^{i_m}$ be a monomial in the extended set arising from $[\alpha]$. Let x_k be a non-multiplier of β. Then βx_k, as a multiple of some monomial in $[\alpha]$, is the product of a monomial γ in the extended set by multipliers of γ (§ 101).

We shall prove that γ is higher than β. Let $\gamma = x_1^{j_1} \cdots x_m^{j_m}$. If $j_1 < i_1$, x_1 cannot be a multiplier for γ since j_1 is certainly not the maximum of the degrees in x_1 of the monomials in $[\alpha]$. Hence $j_1 \geqq i_1$. It remains to examine the case in which $j_1 = i_1$. When we dissect f_{i_1}, we find that if $j_2 < i_2$, x_2 can-

not be a multiplier for $x_2^{j_2} \cdots x_m^{j_m}$. Hence $j_2 \geqq i_2$ and we have to study the case in which $j_2 = i_2$. Continuing, we see that γ is not lower than β so that, since $\gamma \neq \beta$, γ is higher than β.

104. We associate with every monomial $x_1^{j_1} \cdots x_m^{j_m}$ the differential operator

$$\frac{\partial^{j_1+\cdots+j_m}}{\partial x_1^{j_1} \cdots \partial x_m^{j_m}}.$$

Then the product of two operators corresponds to the product of the corresponding monomials.

Consider any monomial $\beta = x_1^{i_1} \cdots x_m^{i_m}$ in (7). Let the corresponding differentiation be performed upon u, and after the differentiation, let the non-multipliers of β be given zero values. Every term in the expansion of u which is not divisible by β will disappear during the differentiation. Any term divisible by β whose quotient by β contains non-multipliers of β will disappear when the non-multipliers are made zero. Hence the above operation gives identical results when applied to u and to $\beta f_{i_1 \cdots i_m}$.

105. We study, for its instructive value, rather than for purposes of application, a special system of equations. There will be as many equations as there are monomials $x_1^{i_1} \cdots x_m^{i_m}$ in (7). (A set $[\alpha]$ is supposed to be given.) With each $\beta = x_1^{i_1} \cdots x_m^{i_m}$ in (7), we associate an equation

$$\frac{\partial^{i_1+\cdots+i_m} u}{\partial x_1^{i_1} \cdots \partial x_m^{i_m}} = g_{i_1 \cdots i_m} \tag{8}$$

where $g_{i_1 \cdots i_m}$ is an arbitrarily assigned function of the multipliers of β, analytic for small values of the multipliers.*

We shall show that there is one and only one function u, analytic for $x_i = 0$, $i = 1, \cdots, m$, such that each equation (8) is satisfied, when the non-multipliers of the associated β are zero, for small values of the multipliers. We are not finding an actual solution of the system (8). We are finding

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* If $\beta = 1$, the first member of (8) is u.

a u which satisfies each equation on a spread associated with that equation.

Consider any particular equation (8). Let the second member be integrated i_1 times in succession with respect to x_1 from 0 to x_1, then i_2 times in succession with respect to x_2 from 0 to x_2, and so on. The result will be a function

$$x_1^{i_1} \cdots x_m^{i_m} f_{i_1 \cdots i_m} \tag{9}$$

where $f_{i_1 \cdots i_m}$ is a function of the multipliers of β, analytic for small values of its variables. The function (9) satisfies identically its associated equation in (8). Let u be the sum of all functions (9) obtained from (8). The expression for u as a sum gives the dissection (7) of u relative to $[\alpha]$. Then, by § 104, u satisfies each equation (8) on the spread associated with that equation. If u_1 is a second solution of the problem, it is seen, from § 104, that in the dissection of $u_1 - u$ relative to $[\alpha]$, the series $f_{i_1 \cdots i_m}$ are all zero. Hence u is unique.

Marks

106. Let $y_1, \cdots, y_n$ be analytic functions of $x_1, \cdots, x_m$. Riquier effects an ordering of the y_i and their partial derivatives in the following way.

Let s be any positive integer. We associate with each x_i any ordered set of s non-negative integers

$$u_{i1}, \cdots, u_{is} \tag{10}$$

in which the first integer, u_{i1}, is unity. With each y_i, we associate any ordered set of non-negative integers

$$v_{i1}, \cdots, v_{is}, \tag{11}$$

taking care that y_i and y_j with $i \neq j$ do not have identical sets (11). The jth integer in (10) is called the jth *mark* of x_i, and the jth integer in (11), the jth *mark* of y_i.

If

$$w = \frac{\partial^{k_1 + \cdots + k_m}}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}} y_i \tag{12}$$

we define the jth *mark* of w, $j=1,\cdots,s$ to be $v_{ij}+k_1u_{1j}+\cdots+k_mu_{mj}$.

Consider all of the derivatives of all y_i.* Let w_1 and w_2 be any two of these derivatives. Let the marks of w_1 and w_2 be

$$a_1,\cdots,a_s;\quad b_1,\cdots,b_s$$

respectively. Suppose that the two sets of marks are not identical. We shall say that w_1 is *higher* than w_2 or is *lower* than w_2 according as the first non-zero difference a_i-b_i is positive or is negative. If the two sets of marks are identical, no relation of order is established between w_1 and w_2.

If w_1 is higher than w_2, $\partial w_1/\partial x_i$ is higher than $\partial w_2/\partial x_i$. Also, $\partial w/\partial x_i$ is always higher than w.

When the marks in (10) and (11) are such that a difference in order exists between any two distinct derivatives, the derivatives of the y_i are said to be *completely ordered.*

Suppose that the ordering is not complete. We shall show how to adjoin new marks, after u_{is} and v_{is}, so as to effect a complete ordering. Clearly, the adjunction of such new marks will not disturb any order relationships which may already exist.

Let m additional marks be assigned, as in the following table:

$$\begin{array}{lccccccccc} & x_1 & x_2 & \cdots & x_m & & y_1 & y_2 & \cdots & y_n \\ s+1 & 1 & 0 & \cdots & 0, & & 0 & 0 & \cdots & 0, \\ s+2 & 0 & 1 & \cdots & 0, & & 0 & 0 & \cdots & 0, \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ s+m & 0 & 0 & \cdots & 1, & & 0 & 0 & \cdots & 0. \end{array}$$

Now, let w_1 and w_2 be two derivatives with the same set of $s+m$ marks. The $(s+i)$th mark of w_1 or w_2, $i=1,\cdots,m$, is the number of differentiations with respect to x_i in w_1 or w_2. Hence the same differentiations are effected in w_1 as in w_2. From the definition of the marks of w_1 and w_2, it follows now that the functions of which w_1 and w_2 are

* Each y_i will be considered as a derivative of zero order of itself.

derivatives have the same sets (11). Thus w_1 and w_2 are identical, so that the new ordering is complete.

In everything which follows, we shall deal only with complete orderings.

107. Let $\xi_1, \cdots, \xi_m$; $\zeta_1, \cdots, \zeta_n$ be variables. We associate with w, in (12) the monomial $\xi_1^{k_1} \cdots \xi_m^{k_m} \zeta_i$.

Let $w_1, \cdots, w_t$ be any finite number of distinct derivatives of the y_i. Let the monomial associated above with w_i, $i = 1, \cdots, t$, be α_i. Let g be any positive number. We shall show how to assign, to the ξ_i, ζ_i, real values, not less than unity, in such a way that, if w_i is higher than w_j, we have, for the assigned values, $\alpha_i > g\alpha_j$.

We introduce s new variables $z_1, \cdots, z_s$. With each ξ_i we associate the monomial $z_1^{u_{i1}} \cdots z_s^{u_{is}}$ where the u_{ij} are the marks of x_i. With each ζ_i we associate $z_1^{v_{i1}} \cdots z_s^{v_{is}}$ where the v_{ij} are the marks of y_i. Then each α_i goes over into a monomial $\beta_i = z_1^{a_1} \cdots z_s^{a_s}$ with a_j the jth mark of w_i.

It will evidently suffice to prove that we can attribute to the z_i real values not less than unity in such a way that $\beta_i > g\beta_j$ if w_i is higher than w_j.

Let r be the maximum of the degrees (total) of the β_i. Let k be any positive number, greater than unity and greater than g. We put

$$(13) \qquad z_i = k^{(rs+1)^{s-i}}, \quad i = 1, \cdots, s.$$

Then, if

$$\beta_i = z_1^{a_1} \cdots z_{h-1}^{a_{h-1}} z_h^{a_h} \cdots z_s^{a_s},$$

$$\beta_j = z_1^{a_1} \cdots z_{h-1}^{a_{h-1}} z_h^{b_h} \cdots z_s^{b_s}$$

with $a_h > b_h$, we have, for (13),

$$\frac{\beta_i}{\beta_j} \geqq \frac{z_h}{(z_{h+1} \cdots z_s)^r} \geqq \frac{k^{(rs+1)^{s-h}}}{k^{r(s-h)(rs+1)^{s-h-1}}} \geqq k > g.$$

ORTHONOMIC SYSTEMS

108. Let $y_1, \cdots, y_n$ be unknown functions of $x_1, \cdots, x_m$, whose derivatives have been completely ordered by marks. We consider a finite system σ of differential equations,

$$\frac{\partial^{i_1+\cdots+i_m} y_j}{\partial x_1^{i_1} \cdots \partial x_m^{i_m}} = g_{i_1 \cdots i_m, j} \tag{14}$$

where

(a) *in each equation, g is a function of $x_1, \cdots, x_m$ and of a certain number of derivatives of the y_i, every derivative in g being lower than the first member of the equation;*

(b) *the first members of any two equations are distinct;*

(c) *if w is a first member of some equation, no derivative of w appears in the second member of any equation;*

(d) *the functions g are all analytic at some point in the space of the arguments involved in all of them.**

We do not assume that every y_i appears in a first member.

Riquier calls such a system of equations *orthonomic.*

The derivatives of the y_i which are derivatives of first members in the orthonomic system are called *principal* derivatives. All other derivatives are called *parametric* derivatives.

109. Given an orthonomic system, σ, we shall show how to obtain an orthonomic system with the same solutions, in which, for each y_i appearing in the first members, the monomials corresponding to those first members which are derivatives of y_i form a complete set (§ 102).

Let equations be adjoined to (14), by differentiating the equations in (14), so that, for each y_i which occurs in some first member, the monomials corresponding to the enlarged set of first members constitute a complete set. By § 102, this can be done. We obtain thus a system σ_1 of equations. Certain first members in σ_1 may be obtainable from more than one of the first members in σ. In that case, we use any one of the first members in σ which is available.

Consider any one of the equations in σ. Let w represent its first member, and v the highest derivative in the second member. If we differentiate the equation with respect to x_i, the first member becomes $\partial w / \partial x_i$. The highest deri-

* Thus, in (d), derivatives not effectively present in a g may be regarded as arguments in that g. This does not conflict with (a), in which the arguments considered are supposed to be effectively present.

vative in the new second member will be $\partial v / \partial x_i$, which is lower than $\partial w / \partial x_i$ (§ 106).

It is clear on this basis, that σ_1 satisfies condition (a).

We attend now to (c). Let $\mathfrak{C}$ be an open region in the space of the arguments in the second members in σ in which the second members are analytic. We consider those solutions of σ for which the indicated arguments lie in $\mathfrak{C}$.

The second members in σ_1 may involve derivatives not in the second members in σ. The second members in σ_1 will be polynomials in the new derivatives, with coefficients analytic in $\mathfrak{C}$.

Let w be the highest derivative present in a second member in σ_1 which is a derivative of a first member in σ_1. Then w is not present in any second member in σ, so that it appears rationally and integrally in the second members in σ_1. Let w be a derivative of v, the first member of the equation $v = g$ in σ_1. Then w can be replaced, in the second members in σ_1, by its expression obtained on differentiating g. We obtain thus a system σ_2 with the same solutions as σ_1 (or σ), and with the same first members as σ_1. The system σ_2 satisfies condition (a). The derivatives higher than w which appear in the second members in σ_2 also appear in the second members in σ_1. Hence, if w_1, present in the second members in σ_2, is a derivative of a first member in σ_2 then w_1 is lower than w. We treat w_1 as w was treated. Since there cannot be an infinite sequence of derivatives each lower than the preceding one, we must arrive, in a finite number of steps, at a system τ, with the same solutions as σ, which satisfies (a), (b), (c), and which has complete sets of monomials corresponding to its first members. The second members in τ will be polynomials in any derivatives not present in the second members of σ. Hence assumption (d) is satisfied for $\mathfrak{C}$ and for any values of the new derivatives. Thus τ is orthonomic and has the same solutions as σ.*

* With the values of the arguments in the second members in σ lying in $\mathfrak{C}$.

Of course, whether we employ σ or τ, we get the same set of principal derivatives and the same parametric derivatives.

110. We consider an orthonomic system, σ, whose first members, as in § 109, yield complete sets of monomials. We are going to seek solutions of σ, analytic at some point, which, with no loss of generality, may be taken as $x_i = 0$, $i = 1, \cdots, m$.

Consider any y_i. Let numerical values be assigned to the parametric derivatives of y_i, at the origin, with the sole conditions that the second members in σ are analytic for the values given to the derivatives in them and that the series

$$\sum \frac{a_{j_1 \cdots j_m}}{j_1! \cdots j_m!} x_1^{j_1} \cdots x_m^{j_m} \tag{15}$$

where the a are the values of the parametric derivatives, the subscripts indicating the type of differentiation, converges in a neighborhood of the origin. The series (15) is called the *initial determination* of y_i. If y_i does not appear in a first member, (15) is a complete Taylor series.

In what follows, we suppose an initial determination to be given for each y_i. We shall then develop a process for calculating the values of the principal derivatives at the origin. There will result analytic functions y_i which satisfy each equation of σ on the spread obtained by equating to zero the non-multipliers of the monomial corresponding to the first member. Later we shall obtain a condition for the y_i to give an actual solution of σ.

In the dissection (7) of each y_i which we shall obtain,* those terms whose monomials are multiples of monomials in the complementary set will constitute the initial determination of y_i. Thus the initial determination of each y_i is a linear combination of a certain number of arbitrary functions, with monomials for coefficients, the variables in the arbitrary functions being specified. This description of the degree of

* This dissection is based on the complete set of monomials corresponding to y_i.

generality of the solution of a system of equations is one of the most important aspects of Riquier's work.

We replace each y_i which does not figure in any first member in σ by an arbitrarily selected initial determination. Then σ becomes an orthonomic system in the remaining y_i, with the same principal derivatives as before for the remaining y_i. On this basis, we assume, with no loss of generality, that *every y_i figures in a first member.*

111. We use the symbol δ to represent differential operators. Any principal derivative, δy_i, which is not a first member in σ, can be obtained from one and only one first member in σ by differentiation with respect to multipliers of the monomial corresponding to that first member. This is because the first members yield complete sets. We have thus a unique expression for δy_i,

$$\delta y_i = g, \tag{16}$$

where the derivatives in g are lower than δy_i.

The infinite system obtained by adjoining all equations (16) to σ will be called τ. Let p be any non-negative integer. The systems of equations in τ whose first members have p for first mark will be called τ_p. Since the first mark of a derivative is the sum of the order of the derivative and of the first mark of the function differentiated, each τ_p has only a finite number of equations.

Let a be the minimum, and b the maximum, of the first marks in the first members in σ. For the values assigned, in § 110, to the parametric derivatives, the equations $\tau_a, \tau_{a+1}, \cdots, \tau_b$ determine uniquely the values at the origin of the principal derivatives whose first mark does not exceed b. In short, the lowest such derivative has an equation which determines it in terms of parametric derivatives; the principal derivative next in ascending order is determined in terms of parametric derivatives, and, perhaps, the first principal derivative, and so on.

We subject the unknowns y_j to the transformation

$$y_j = \bar{y}_j + \varphi_j + \sum \frac{c_{j,i_1\cdots i_m}}{i_1! \cdots i_m!} x_1^{i_1} \cdots x_m^{i_m} \tag{17}$$

where φ_j is the chosen initial determination of y_j and where the c are the principal derivatives at the origin of y_j, of first mark not exceeding b, found as above.

Then σ goes over into a system σ' in the $\bar{y}_j$. In the new system, we transpose the known terms in the first members (these come from the known terms in (17)) to the right. The new system will be orthonomic in the $\bar{y}_j$, with the same monomials for its first members as in σ. The second members will be analytic when each x_i and each parametric derivative is small.

The system τ' for σ', analogous to τ for σ, is obtained by executing the transformation (17) on the equations of τ.

Thus, if we give to the $\bar{y}_i$, in σ', initial determinations which are identically zero, the principal derivatives at the origin, of first mark not exceeding b, will be determined as zero by $\tau'_a, \cdots, \tau'_b$.

On this account, we limit ourselves, without loss of generality, to the search of solutions $y_1, \cdots, y_n$, of σ, with initial determinations identically zero, assuming that the system $\tau_a, \cdots, \tau_b$ yields zero values at the origin for the principal derivatives whose first marks do not exceed b.

112. In the second members in τ_{b+1}, no derivatives appear whose first marks exceed $b+1$. Those derivatives whose first marks are $b+1$ enter linearly, because they come from the differentiation of derivatives of first mark b in τ_b.

We denote by $\delta_k y_i$ the second member of (12). Then every equation in τ_{b+1} is of the form

$$\delta_i y_\alpha = \sum p_{i\alpha j\beta}\, \delta_j y_\beta + q_{i\alpha} \tag{18}$$

where the $\delta_j y_\beta$ are of first mark $b+1$ and where the p and q involve the x_i and derivatives whose first marks are b or less.

In (18), we consider every derivative of first mark $b+1$ which is lower than $\delta_i y_\alpha$ to be present in the second member. If necessary, we take $p_{i\alpha j\beta} = 0$.

Consider any $\delta_i y_\alpha$ in (18). Suppose that there is a β such that y_β has derivatives of first mark $b+1$ which are lower than $\delta_i y_\alpha$. For every such β, we let $r_{i\alpha\beta}$ represent the

number of derivatives of y_β, of first mark $b+1$, which are lower than $\delta_i y_\alpha$. For every other β, we let $r_{i\alpha\beta}=1$, and we suppose that a single derivative of y_β of first mark $b+1$ appears in the second member of (18), with a zero coefficient. We can thus not continue to say that every derivative in the second member of (18) is lower than the first member, but no difficulty will arise out of this; only a question of language is involved.

Let r be the maximum of the $r_{i\alpha\beta}$.

The p and q in (18) are analytic for small values of their arguments. Let the p and q be expanded as series of powers of their arguments.

Let $\varepsilon>0$ be such that each of the above series converges for values of its arguments which all exceed ε in modulus. Let $h>0$ be such that each p and each q has a modulus less than h when the arguments do not exceed ε.

Let λ be any positive number less than $1/n$.

Following § 107, we determine positive numbers ξ_i, ζ_i, not less than unity such that, if $\delta_i y_\alpha$ and $\delta_j y_\beta$ are of first mark $b+1$, with $\delta_i y_\alpha$ higher than $\delta_j y_\beta$, we have

$$\frac{\xi_1^{i_1}\cdots\xi_m^{i_m}\zeta_\alpha}{\xi_1^{j_1}\cdots\xi_m^{j_m}\zeta_\beta}>\frac{h\,r}{\lambda}. \tag{19}$$

In what follows, we associate with each y_i a new unknown function u_i.

Let

$$\varrho=\frac{\xi_1 x_1+\cdots+\xi_m x_m+\sum\delta u}{\varepsilon}$$

where Σ ranges over all derivatives of $u_1,\cdots,u_n$ whose first mark does not exceed b ($\delta_i u_\alpha$ is supposed to have the same marks as $\delta_i y_\alpha$).

We consider the system of equations

$$\begin{aligned}\delta_i u_\alpha=&\frac{1}{(1-\varrho)}\sum\frac{\lambda}{r_{i\alpha\beta}}\,\frac{\xi_1^{i_1}\cdots\xi_m^{i_m}\zeta_\alpha}{\xi_1^{j_1}\cdots\xi_m^{j_m}\zeta_\beta}\,\delta_j u_\beta\\&+\frac{h\,\xi_1^{i_1}\cdots\xi_m^{i_m}\zeta_\alpha}{1-\varrho}\end{aligned} \tag{20}$$

which has the general form of (18), with alterations of the form of the p and q.

The function

$$\frac{h}{1 - \dfrac{x_1 + \cdots + x_m + \sum \delta u}{\varepsilon}}$$

is a majorant for every p and every q. As each ξ_i is at least unity, the same is true of $h/(1-\varrho)$.

Thus, in virtue of (19), wherever a $\delta_j y_\beta$ is lower than $\delta_i y_\alpha$ in an equation in (18), the coefficient of $\delta_j u_\beta$ in the corresponding equation of (20) will be a majorant for the coefficient of $\delta_j y_\beta$. In the exceptional case where a $\delta_j y_\beta$ is not lower than $\delta_i y_\alpha$ and thus has a zero coefficient, the corresponding coefficient in (20) is certainly a majorant. Evidently the terms in (20) which correspond to the q in (18) are majorants of the q.

113. We shall show that (20) has a solution in which each u_i is a function of

$$\xi_1 x_1 + \cdots + \xi_m x_m. \tag{21}$$

Consider, in (20), all derivatives of a particular u_α whose first marks are $b+1$. The first mark of any such derivative is the order (total) of the derivative, plus the first mark of u_α. Hence all of the derivatives of u_α which are of first mark $b+1$ are of the same order, say g_α.

Let the u_α, in what follows, represent functions of (21). Put $u_\alpha = \zeta_\alpha u'_\alpha$ and let $u'_{\alpha i}$ be the $(i_1 + \cdots + i_m)$th derivative of u'_α with respect to (21). Then

$$\frac{\partial^{i_1+\cdots+i_m}}{\partial x_1^{i_1} \cdots \partial x_m^{i_m}} u_\alpha = \xi_1^{i_1} \cdots \xi_m^{i_m} \zeta_\alpha u'_{\alpha i}.$$

When the u_α are functions of (21), ϱ becomes a function ϱ' of (21) and of the derivatives of the u'_α of order less than g_α, $\alpha = 1, \cdots, n$. Equations (20) reduce to

$$u'_{\alpha g_\alpha} = \lambda \sum_{\beta=1}^{n} \frac{1}{1-\varrho'} u'_{\beta g_\beta} + \frac{h}{1-\varrho'}. \tag{22}$$

There will be n equations in (22), one for each α. All equations in (20) in which a given u_α appears in the first member yield the same equation (22). We write (22) as

$$u'_{\alpha g_\alpha} = \varrho' u'_{\alpha g_\alpha} + \lambda \sum_{\beta=1}^{n} u'_{\beta g_\beta} + h. \tag{23}$$

When (21) is zero and when the $u'_{\alpha i}$, $i = 0, \cdots, g_\alpha - 1$, for each α, are given zero values, the determinant of (22) with respect to the $u'_{\alpha g_\alpha}$ is

$$\begin{vmatrix} 1-\lambda, & -\lambda, & \cdots, & -\lambda \\ -\lambda, & 1-\lambda, & \cdots, & -\lambda \\ \cdot & \cdot & \cdot & \cdot \\ -\lambda, & -\lambda, & \cdots, & 1-\lambda \end{vmatrix}$$

This determinant is not zero. In short, the equations

$$\begin{array}{c} (1-\lambda) z_1 - \lambda z_2 - \cdots - \lambda z_n = c_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ -\lambda z_1 - \lambda z_2 - \cdots + (1-\lambda) z_n = c_n \end{array} \tag{24}$$

imply

$$(1 - n\lambda) z_i = \lambda(c_1 + \cdots + c_n) + (1 - n\lambda) c_i$$

so that the determinant cannot vanish for $\lambda < 1/n$.*

Then the $u'_{\alpha g_\alpha}$ can be expressed as functions of the other quantities in (23), analytic when the arguments are small. By the existence theorem for ordinary differential equations, (23) has a solution with the $u'_{\alpha i}$ zero, for $i < g_\alpha$, when (21) is zero. The functions in this solution will be analytic for (21) small.

114. We shall prove that, in the solution just found, all $u'_{\alpha i}$ with $i \geqq g_\alpha$ are positive for (21) zero. For (21) zero, we have

$$u'_{\alpha g_\alpha} - \lambda \sum_{\beta=1}^{n} u'_{\beta g_\beta} = h.$$

* For $i = 1$, subtract each equation from the first, in succession, and substitute the results into the first.

Referring to (24), we see that, since $\lambda < 1/n$, the z_i are positive if the c_i are all positive. Then the $u'_{\alpha g_\alpha}$ are positive for every α.

Differentiating (23), we find, for (21) zero,

$$u'_{\alpha, g_\alpha+1} - \lambda \sum_{\beta=1}^{n} u'_{\beta, g_\beta+1} = k_\alpha$$

where the k_α are positive. Again, the solution consists of positive numbers. Continuing, we obtain our result.

What precedes shows that (20) has a solution, analytic at the origin, with every derivative of first mark less than $b+1$ equal to zero and every other derivative positive, at the origin.

115. We now return to the system σ. With the procedure employed, in § 111, for the determination, at the origin, of the principal derivatives of first mark not greater than b, we determine the values of all principal derivatives at the origin. We can ascend, step by step, through all the principal derivatives, because each τ_p in § 111 has only a finite number of equations.

We obtain thus a complete power series for each y_i. We are going to prove that these power series converge for small values of the x_i.

Let $\delta_i y_\alpha$ be any principal derivative. We shall prove that the modulus of this derivative at the origin does not exceed the value at the origin found for $\delta_i u_\alpha$ in § 114.

For derivatives of first mark less than $b+1$, this is certainly true; those derivatives have zero values. Let the result hold for all derivatives lower than some $\delta_i y_\gamma$ of first mark greater than b. The equation in τ for $\delta_i y_\gamma$ is either in (18), or is found by differentiating some equation in (18). Consider the corresponding equation for $\delta_i u_\gamma$, which is either in (20), or obtained from (20) by differentiation.

We shall consider the expressions for $\delta_i y_\gamma$ and $\delta_i u_\gamma$ as power series in the x_i and in the derivatives in terms of which $\delta_i y_\gamma$ and $\delta_i u_\gamma$ are expressed.

We see that, for every term in the series for $\delta_i y_\gamma$, there is a dominating term in the series for $\delta_i u_\gamma$. What is more,

the series for $\delta_i u_\gamma$ may have other terms, involving $\delta_i u_\gamma$ itself, or even higher derivatives. This is because of the exceptional terms in (20), introduced in § 112.*

Each term in $\delta_i u_\gamma$ which has a corresponding term in $\delta_i y_\gamma$ is at least as great as the modulus of that term at the origin, for such terms involve only lower derivatives than $\delta_i y_\gamma$ or $\delta_i u_\gamma$. Terms in $\delta_i u_\gamma$ which have no corresponding terms in $\delta_i y_\gamma$ are zero or positive at the origin. They will be positive if they involve no x_i, and contain only derivatives of first mark at least $b+1$ (§ 114). This proves that the value determined for each $\delta_i y_\alpha$ by τ has a modulus not greater than the value at the origin of $\delta_i u_\alpha$.

Thus the series obtained for the y_i converge in a neighborhood of the origin.

116. We shall now see to what extent the analytic functions y_i, just obtained, are solutions of σ.

Consider any equation $\delta y_i = g$ in σ. This equation, and all equations obtained from it by differentiation with respect to multipliers of the monomial corresponding to the first member, are satisfied, at the origin, by the derivatives of $y_1, \cdots, y_n$ at the origin. Hence, if we substitute $y_1, \cdots, y_n$ into $\delta y_i - g$, we obtain a function k of $x_1, \cdots, x_n$ which vanishes at the origin, together with its derivatives with respect to the above multipliers. Thus, in the expansion of k, only non-multipliers occur. Then k vanishes when the non-multipliers are zero.

Hence $y_1, \cdots, y_n$ *satisfy each equation of* σ *on the spread obtained by equating to zero the non-multipliers corresponding to the first member of the equation.*

117. Let us return now to the most general orthonomic system σ whose first members give complete sets of monomials. We do not suppose that every y_i appears in some first member.

We consider any point $x_i = a_i$, $i = 1, \cdots, m$, subject to obvious conditions of analyticity. Let any values be given to the parametric derivatives of the y_i at $a_1, \cdots, a_m$, so as to yield convergent initial determinations. Then the principal

* In our present language, all derivatives in a second member in (18) are lower than the first member.

derivatives are determined uniquely by σ in such a way as to yield analytic functions $y_1, \cdots, y_n$ which satisfy each equation in σ on the spread obtained by equating to a_i each non-multiplier x_i corresponding to the first member of the equation.

This is an immediate consequence of the preceding sections.

PASSIVE ORTHONOMIC SYSTEMS

118. Let σ be an orthonomic system, described as in the preceding section. Let the equations in σ be listed so that their first members form an ascending sequence, and let them be written

$$v_i = 0, \qquad i = 1, \cdots, t. \tag{25}$$

If v_i is $\delta y_j - g$, we attribute to v_i the s marks of δy_j. This establishes order relations among the v_i, according to the convention of § 106. To all of the derivatives of v_i, we attribute marks as in § 106. Thus, the marks of δv_i will be the marks of the highest derivative in δv_i. By the *monomial corresponding to* v_i, we mean the monomial corresponding to δy_j. We shall refer to δy_j as the *first term* in v_i. By the first term of a derivative of v_i, we shall mean the corresponding derivative of δy_j.

Consider a v whose corresponding monomial, α, has non-multipliers. Let x_i be such a non-multiplier. By § 103, $x_i \alpha$ is the product of a β, in the same complete set as α and higher than α, by multipliers of β. Hence, there is a v_p, higher than v, such that some δv_p has the same first term as $\partial v / \partial x_i$. Then, in the expression

$$\frac{\partial v}{\partial x_i} - \delta v_p \tag{26}$$

all derivatives effectively present are lower than the first term of $\partial v / \partial x_i$.

It is clear that (26) is a polynomial in such principal derivatives as it may involve. Let w be the highest such principal derivative. Then w is the first term of some expression δv_q, where δv_q is lower than $\partial v / \partial x_i$. We choose v_q so that w is obtained from it by differentiation with respect

to multipliers of the corresponding monomial. This makes v_q unique. Let then, identically,

$$w = \delta v_q + k, \tag{27}$$

where the derivatives in k are all lower than w. We replace w in (26) by its expression in (27) and find, identically,

$$\frac{\partial v}{\partial x_i} = \delta v_p + h_1(\delta v_q, \cdots),$$

where h_1 is a polynomial in δv_q whose coefficients involve no principal derivative as high as w. Let w_1 be the highest principal derivative in h_1. We give it the treatment accorded to w and find

$$\frac{\partial v}{\partial x_i} = \delta v_p + h_2(\delta v_q, \delta v_r, \cdots),$$

where h_2 is a polynomial in δv_q, δv_r. Continuing, we find in a unique manner, an identity

$$\frac{\partial v}{\partial x_i} = \delta v_p + h(\delta v_q, \cdots, \delta v_z), \tag{28}$$

in which the coefficients in h involve only parametric derivatives. We now write (28) in the form

$$\frac{\partial v}{\partial x_i} = \delta v_p + \gamma(\delta v_q, \cdots, \delta v_z) + \mu \tag{29}$$

where μ is the term of zero degree in h. Then μ is an expression in the parametric derivatives alone. The expression γ vanishes when $\delta v_q, \cdots, \delta v_z$ are replaced by 0.

It is clear that, for any solution of σ, we must have $\mu = 0$. The totality of equations $\mu = 0$, obtained from all equations of σ for which the monomial corresponding to the first member has non-multipliers, all non-multipliers being used, are called the *integrability conditions* for σ.

119. If every expression μ is identically zero, the system σ is said to be *passive.*

We shall prove that, if σ is passive, the n functions $y_1, \cdots, y_n$, described in § 117, which satisfy each equation in σ on a certain spread, constitute an actual solution of σ.

What we have to show is, that for these functions, every v_i in (25) vanishes identically.

When the y_j above are substituted into v_i, we obtain a function u_i of $x_1, \cdots, x_m$. If v_i has no non-multipliers, $u_i = 0$. Otherwise, u_i vanishes when the non-multipliers of the monomial corresponding to v_i are equated to their a_i.

If, in (29), where μ is now identically zero, the parametric derivatives in γ are replaced by their expressions as functions of the x_i, found from the y_i, (29) becomes a system φ of differential equations in the *unknowns* v_i. Since (29) consisted of identities, before these replacements, φ is satisfied by $v_i = u_i$, $i = 1, \cdots, t$.

We now attribute to each x_i an additional mark 0, and to each v_i an additional mark $t - i$. With this change, the derivatives of the v_i will be completely ordered and the first member in each equation in φ will be higher than every derivative in the second member.

If the second members in φ contain derivatives of the first members, we can get rid of such derivatives, step by step. Then φ goes over into an orthonomic system ψ, with the same first members as φ.

For our purposes, it is unnecessary to adjoin new equations to ψ as in § 109. Consider any unknown v_i which appears in a first member. The derivatives of v_i in the first members will be taken with respect to certain variables

$$x_a, \cdots, x_d. \tag{30}$$

The variables (30) when equated to their a_i, give a spread on which u_i vanishes.

The parametric derivatives of v_i will be the derivatives taken with respect to the variables not in (30). For the corresponding u_i, each of these parametric derivatives is zero. Now we know that, for given values of the parametric derivatives, there is at most one solution of ψ. But $v_i = 0$, $i = 1, \cdots, t$ is a solution of ψ for which all parametric derivatives vanish. Hence $u_i = 0$, $i = 1, \cdots, t$.

This proves that, *given a passive orthonomic system, there is one and only one solution of the system for any given initial determinations.*

CHAPTER X

SYSTEMS OF ALGEBRAIC PARTIAL DIFFERENTIAL EQUATIONS

DECOMPOSITION OF A SYSTEM INTO IRREDUCIBLE SYSTEMS

120. We consider n unknown functions, $y_1, \cdots, y_n$, of m independent variables, $x_1, \cdots, x_m$. Definitions will usually be as for the case of one independent variable, and will be given, formally, only when there is some necessity for it.

We assume marks to have been assigned to the x_i and y_i in such a way as to order completely the derivatives of the y_i.

By a *form*, we shall mean a polynomial in the y_i and any number of their partial derivatives, with coefficients which are functions of the x_i, meromorphic at each point of a given open region $\mathfrak{A}$ in the space of the x_i. By a *field*, we shall mean a set of functions meromorphic at each point of $\mathfrak{A}$, the set being closed with respect to rational operations and partial differentiation. We assume a field $\mathfrak{F}$ to be given in advance. Where the contrary is not stated, the coefficients in a form will belong to $\mathfrak{F}$.

121. By the *leader* of a form A which actually involves unknowns, we shall mean the highest derivative present in A.

Let A_1 and A_2 be two forms which actually involve unknowns. If A_2 has a higher leader than A_1, then A_2 will be said to be of *higher rank* than A_1. If A_1 and A_2 have the same leader, and if the degree of A_2 in the common leader exceeds that of A_1, then again, A_2 will be said to be of higher rank than A_1. A form which effectively involves unknowns will be said to be of higher rank than a form which does not.

Two forms, for which no difference in rank is created by what precedes, will be said to be of the same rank.*

The lemma of § 2 goes over immediately to the case of several variables.

122. If A_1 involves unknowns, A_2 will be said to be *reduced with respect to* A_1 if A_2 contains no derivative (proper) of the leader of A_1 and if A_2 is of lower degree than A_1 in the leader of A_1. A set of forms

$$(1) \qquad A_1,\ A_2,\ \cdots,\ A_r$$

will be called an *ascending set* if either

(a) $r = 1$ *and* $A_1 \neq 0$, or

(b) $r > 1$, A_1 *involves unknowns and, for* $j > i$, A_j *is of higher rank than* A_i *and reduced with respect to* A_i.

When (b) holds, the leader of A_j is higher than that of A_i for $j > i$.

Relative rank for ascending sets is defined exactly as in § 3. If Φ_1, Φ_2, Φ_3 are ascending sets with $\Phi_1 > \Phi_2$ and $\Phi_2 > \Phi_3$ then $\Phi_1 > \Phi_3$. We prove the following lemma.

LEMMA. *Let*

$$(2) \qquad \Phi_1,\ \Phi_2,\ \cdots,\ \Phi_q,\ \cdots$$

be an infinite sequence of ascending sets such that Φ_{q+1} *is not higher than* Φ_q *for any* q. *Then there exists a subscript* r *such that, for* $q > r$, Φ_q *has the same rank as* Φ_r.

For q large, the first forms in the Φ_q have the same rank. These first forms will either be free of the unknowns, or else will have the same leader, say p_1. We have only to consider the latter possibility, and may limit ourselves to the case in which Φ_q with q large has at least two forms. For q large, the second forms will have the same leader, say p_2. Now p_2 is not a proper derivative of p_1. As we saw above, p_2 is higher than p_1. We may confine ourselves now to the case in which Φ_q, for q large, has at least three

* It will be noticed that the above definitions of relative rank do not specialize into those of § 2. This is due to the fact that the first mark of each x_i is unity.

forms. Then, for q large, the third forms will all have the same leader, p_3, higher than p_1 and p_2 and not a derivative of either of them. Thus, our result holds, unless there is an infinite sequence

$$p_1, p_2, \cdots, p_q, \cdots$$

of derivatives which increase steadily in rank, no p_q being a derivative of a p_i with $i < q$. But this contradicts Riquier's theorem on sequences of monomials proved in § 99.

On the basis of the above lemma, we define a *basic set* of a system Σ which contains non-zero forms, to be an ascending set of Σ of least rank.

If A_1 in (1) involves unknowns, a form F will be said to be *reduced with respect to* (1) if F is reduced with respect to A_i, $i = 1, \cdots, r$.

Let Σ be a system for which (1), with A_1 not free of the unknowns, is a basic set. Then no non-zero form of Σ can be reduced with respect to (1). If a non-zero form, reduced with respect to (1), is adjoined to Σ, the basic sets of the resulting system are lower than (1).

123. In this section, we deal with an ascending set (1) in which A_1 involves unknowns.

If a form G has a leader, p, we shall call the form $\partial G / \partial p$ the *separant* of G. The coefficient of the highest power of p in G will be called the *initial* of G.

Let S_i and I_i be, respectively, the separant and initial of A_i in (1).

We shall prove the following result.

Let G be any form. There exist non-negative integers, s_i, t_i, $i = 1, \cdots, r$, such that, when a suitable linear combination of the A_i and a certain number of their derivatives, with forms for coefficients, is subtracted from

$$S_1^{s_1} \cdots S_r^{s_r} I_1^{t_1} \cdots I_r^{t_r} G,$$

the remainder, R, is reduced with respect to (1).

Let p_i be the leader of A_i. We limit ourselves, as we may, to the case in which G involves derivatives, proper or

improper, of the p_i. Let the highest derivative in G which is a derivative of a p_i be q and let q be a derivative of p_j. For the sake of uniqueness, if there are several possibilities for j, we use the largest j available. To fix our ideas, we assume q higher than p_r. Then

$$S_j^g G = CA_j' + B$$

where A_j' is a derivative of A_j with q for leader, and where B is free of q. Because A_j' and S_j involve no derivative higher than q, B involves no derivative of a p_i which is as high as q. For uniqueness we take g as small as possible.

If B involves a derivative of a p_i which is higher than p_r, we give B the treatment accorded to G. After a finite number of steps we arrive at a unique form D which differs by a linear combination of derivatives of the A_i from a form

$$S_1^{g_1} \cdots S_r^{g_r} G.$$

The form D involves no derivative of a p_i which is higher than p_r.

We then find a relation

$$I_r^{t_r} D = HA_r + K$$

where K is reduced with respect to A_r. The form K may involve p_r. Aside from p_r, the only derivatives of the p_i present in K are derivatives of $p_1, \cdots, p_{r-1}$. Such derivatives are lower than p_r. Let q_1 be the highest of them.

Suppose that q_1 is higher than p_{r-1}. We give K the treatment accorded to G. In a finite number of steps, we arrive at a unique form L which differs from some

$$S_1^{h_1} \cdots S_{r-1}^{h_{r-1}} I_{r-1}^{t_{r-1}} K$$

by a linear combination of A_{r-1} and the derivatives of $A_1, \cdots, A_{r-1}$. The form L is reduced with respect to A_r and A_{r-1}. Aside from p_r and p_{r-1}, the derivatives of the p_i in L are derivatives of $p_1, \cdots, p_{r-2}$, and all such derivatives are lower than p_{r-1}.

Continuing, we determine, in a unique manner, a form R as described in the statement of the lemma. We call R the *remainder of G with respect to* (1).

124. The argument of §§ 7–10 now goes over, without change, to the case of several independent variables. We secure the lemma:

LEMMA. *Every infinite system of forms in $y_1, \cdots, y_n$ has a finite subsystem whose manifold is identical with that of the infinite system.**

As in § 13, we prove the

THEOREM. *Every system of forms in $y_1, \cdots, y_n$ is equivalent to a finite number of irreducible systems.*

The decomposition is unique in the sense of § 14.

As an example, we consider the equation

$$z-(px+qy)+p^2+q^2 = 0, \tag{3}$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$. Differentiating with respect to x, we find

$$-(rx+sy)+2(pr+qs) = 0 \tag{4}$$

where $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\, \partial y$, $t = \partial^2 z/\partial y^2$. Differentiating (3) with respect to y, we find

$$-(sx+ty)+2(ps+qt) = 0. \tag{5}$$

* This lemma is very different from, and is not to be confused with, the theorem of Tresse for general (non-algebraic) systems of partial differential equations. (Acta Mathematica, vol. 18, (1894), p. 4.) In using the implicit function theorem to solve his system for certain derivatives, Tresse has necessarily to confine himself to a portion of the manifold of his system. In fact, it is not easy to imagine systems other than linear systems for which Tresse's argument and result have a definite meaning. On this basis, the above lemma, together with the theorem of § 129, may be regarded as an extension, to general algebraic systems, of Tresse's result, as applied to linear systems. Thus the relation between Tresse's theorem and our lemma is quite like that between the theorem that a system of linear functions of n variables contains at most $n+1$ linearly independent functions, and Hilbert's theorem on the existence of a finite basis for any system of polynomials in n variables. The use of Riquier's theorem of § 99 was suggested to us by what is contained in Tresse's work. This is the only common feature of the two arguments.

From (4) and (5), we obtain

$$(rt - s^2)(x - 2p) = 0; \qquad (rt - s^2)(y - 2q) = 0.$$

Thus, either $rt - s^2 = 0$ or $z = (x^2 + y^2)/4$. The latter solution of (3) does not annul $rt - s^2$. Thus (3) is a reducible system. As one can see from what follows, it is equivalent to two irreducible systems.

Basic sets of closed irreducible systems

125. Let Σ be a non-trivial closed irreducible system for which

(6) $$A_1, A_2, \cdots, A_r,$$

is a basic set. A solution of (6) for which no separant or initial vanishes will be called a *regular solution* of (6). Evidently such regular solutions exist. The remainder, with respect to (6), of any form of Σ, is zero. Hence, every regular solution of (6) is a solution of Σ. Furthermore, Σ consists of all forms which vanish for the regular solutions of (6).

We represent by $\xi_1, \cdots, \xi_m$, more briefly by ξ, a point in $\mathfrak{A}$ at which the coefficients in (6) are analytic. We use the symbol $[\eta]$ to designate any set of numerical values which one may choose to associate with the derivatives appearing in (6). The existence of regular solutions of (6) guarantees the existence of a set ξ, $[\eta]$ for which every A_i vanishes, but for which no separant or initial vanishes. In what follows, we deal with such a set.

Let p_i be the leader of A_i. The equation $A_1 = 0$, treated as an algebraic equation for p_1, determines p_1 as a function of the x_i and the derivatives lower than p_1 in A_1, the function being analytic for x_i close to ξ_i and for the derivatives lower than p_1 close to their values among the $[\eta]$. The value of the function p_1 for the special arguments stipulated above will be the value for p_1 in $[\eta]$. Let the expression for p_1 be substituted into A_2. We can then solve $A_2 = 0$ for p_2, expressing p_2 as a function of the x_i and of

the derivatives other than p_1 and p_2 appearing in A_1 and A_2. We substitute the expressions for p_1 and p_2 into A_3, solve $A_3 = 0$ for p_3, and continue, in this manner, for all forms in (6).

We will find thus a set of expressions for the p_i, each p_i being given as an analytic function of the x_i and of the derivatives other than $p_1, \cdots, p_r$ in (6). We write

$$(7) \qquad p_i = g_i, \quad i = 1, \cdots, r.$$

If the equations (7) are considered as differential equations for the y_i, they will form an orthonomic system. We shall prove the

THEOREM: *The orthonomic system $p_i = g_i$ is passive.*

As in § 65, we see that if (6) is considered as a set of simple forms in the symbols for the derivatives, (6) will be a basic set of a prime system, Λ.* The unconditioned unknowns in Λ will be those corresponding to the parametric derivatives in (7). We form a simple resolvent for Λ, with

$$(8) \qquad w = b_1 p_1 + \cdots + b_r p_r,$$

the b_i being integers. Let the resolvent be

$$(9) \qquad B_0 w^s + \cdots + B_s = 0,$$

and let the expressions for the p_i be

$$(10) \qquad p_i = \frac{E_{i0} + \cdots + E_{i,s-1} w^{s-1}}{D}, \quad i = 1, \cdots, r,$$

(see § 59), where the B_i, E_{ij} and D are simple forms in the (symbols for the) parametric derivatives. If the parametric derivatives are specialized as functions of the x_i for which $B_0 D \neq 0$, the functions p_i determined by (9) and (10) give (in some open region) all of the solutions of Λ for the given specialization of the parametric derivatives.

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* All results relative to simple forms, which we employ, carry over without difficulty to several variables.

The relations (9) and (10) continue to hold if the p_i are replaced in (8) and (10) by the functions g_i appearing in (7). For, let the arguments in the g_i be given any vulues, close to those in ξ, $[\eta]$, for which $B_0\ D$ does not vanish. Let the value given to x_i be ξ_i'. If the parametric derivatives are held fast at the values just assigned to them, while x_i ranges over the neighborhood of ξ_i', then the p_i in (7) become functions of the x_i, which, with the constant values of the parametric derivatives, give a solution of $\varDelta$. For this solution, $B_0\ D \neq 0$, so that (9) and (10) hold. Now the values of the p_i in this solution, at $x_i = \xi_i'$, are the values of the functions g_i in (7) with the arguments specialized as above. This shows that the g_i can replace the p_i in (8), (9), (10).

We thus consider each g_i in (7) to be expressed by the second member of (10), where w is a function of the x_i and the parametric derivatives, analytic when the arguments are close to their values in ξ, $[\eta]$.

It may be, however, that the expressions (10) are meaningless for the particular values ξ, $[\eta]$; that is D may vanish for those values. To take care of this point, and of a point which will arise later, we pass to values ξ', $[\eta']$, close to ξ, $[\eta]$, for which DB_0K, where K is the discriminant of (9), does not vanish. After we have proved the passivity of (2) for the neighborhood of ξ', $[\eta']$, the passivity for the neighborhood of ξ, $[\eta]$ will follow.

Let equations be adjoined to (7), as in § 109, so as to form an orthonomic system, σ, whose first members give complete sets of monomials. Let us see how the equations in σ can be written. From (10) we find

$$(11)\quad \frac{\partial p_i}{\partial x_j} = \frac{(H_{i0} + \cdots + H_{i,s-1} w^{s-1}) + (J_{i0} + \cdots + J_{i,s-2} w^{s-2})\, \partial w/\partial x_j}{D^s}.$$

Let P represent the first member of (9). Let $Q = \partial P/\partial w$. Then

$$(12) \qquad \frac{\partial w}{\partial x_j} = \frac{U_0 + \cdots + U_{s-1} w^{s-1}}{Q}.$$

Now the resultant of P and Q equals, to within sign, $B_0 K$.* This means that

$$B_0 K = LP + MQ$$

and that the denominator Q in (12) can be replaced by $B_0 K$ (we multiply the numerator by M). In the new expression for $\partial w / \partial x_j$, the degree of the numerator in w may exceed $s-1$. We substitute this new expression for $\partial w / \partial x_j$ into (11). Thus we have

$$(13) \qquad \frac{\partial p_i}{\partial x_j} = \frac{F_0 + \cdots + F_g w^g}{T}$$

where the F_i are free of w, and T is a product of powers of D, B_0, K.

We get expressions similar to the second member of (13) for all derivatives of the p_i. If principal derivatives appear in the F_i, we get rid of them, step by step. At the end, we depress the degrees in w of the numerators in the expressions to less than s. This is accomplished by a division by P; the division introduces a power of B_0 into the denominator.

All in all, each equation in σ will have the form

$$(14) \qquad \delta y = \frac{F_0 + \cdots + F_{s-1} w^{s-1}}{T},$$

where T is a product of powers of D, B_0, K, where the F are simple forms in the parametric derivatives.

If we refer now to § 118, we see that every μ has for the neighborhood of ξ', $[\eta']$, an expression like the second member of (14). To establish the passivity of (7), for the neighborhood of ξ, $[\eta]$, we have to show that every μ, as a function of the x_i and of the parametric derivatives, is identically zero.

---

* Perron, Algebra, vol. 1, p. 225.

The form $DB_0 K$, which involves only parametric derivatives, is reduced with respect to (6) and hence is not in Σ. Consider any regular solution of (6) for which $DB_0 K \neq 0$. Let $\xi_1'', \cdots, \xi_m''$ be a point at which $DB_0 K$ and the separants and initials do not vanish, for this solution. Let $[\eta'']$ represent the set of values, at $\xi_1'', \cdots, \xi_m''$, for this solution, of the derivatives in (6). Let us imagine that we have formed the system (7) for the neighborhood of ξ'', $[\eta'']$. Because the calculation of the expressions for the g_i, in (7), in terms of w, involves only rational operations, the expressions will be the same for ξ'', $[\eta'']$ as for ξ', $[\eta']$. The same is true of the expressions for the μ.

Suppose that the expression for some μ, say μ_1, in terms of w is not identically zero. Let Z be the numerator in the expression for μ_1. Then Z vanishes for the above solution of (6). Because P is irreducible, the resultant W of P and Z with respect to w is not identically zero. As W vanishes for all solutions like the above, W is in Σ. This contradicts the fact that W involves only parametric derivatives.

Thus the expression for μ_1 is identically zero. Then μ_1, as an analytic function, vanishes for the neighborhood of ξ', $[\eta']$.* As ξ', $[\eta']$ is arbitrarily close to ξ, $[\eta]$, μ_1 vanishes for the neighborhood of ξ, $[\eta]$.

This proves the passivity of (7).

126. Let (6), with A_1 not free of the unknowns, be an ascending set. We shall find necessary and sufficient conditions for (6) to be a basic set for a closed irreducible system.

As a first necessary condition, we have the condition that (6), when regarded as a set of simple forms, be a basic set for a prime system.

This implies the existence of r analytic functions g_i, as in (7), which annul the A_i, when substituted for the p_i, without annulling any initial or separant (§ 45).

Let ξ, $[\eta]$ be some set of values, as in § 125, for which

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* Note that not all numbers in ξ' $[\eta']$ are arguments of the μ.

no initial or separant vanishes. A second necessary condition is that the system (7) be passive for the neighborhood of ξ, $[\eta]$.

We shall prove that *if* (6), *considered as a set of simple forms, is a basic set of a prime system, and if* (7) *is passive for a single set* ξ, $[\eta]$, *then* (6) *is a basic set of a closed irreducible system.*

Since (7) is passive for the neighborhood of ξ, $[\eta]$, the μ of § 125 must vanish as analytic functions, for the neighborhood of ξ, $[\eta]$. Hence the expressions of the μ in terms of w, which are valid for the neighborhood of ξ', $[\eta']$, vanish identically.

We conclude that for any set of values ξ, $[\eta]$ *at all* which annul the A_i but no initial or separant, (7) is passive.

The passivity of (7) for ξ, $[\eta]$ as above implies that (6) has regular solutions. We shall prove that the system Σ of forms which vanish for all regular solutions of (6) is an irreducible system of which (6) is a basic set.

Let G and H be such that GH is in Σ. Let G_1 and H_1 be, respectively, the remainders of G and H with respect to (6). There may be, in G_1 and H_1, parametric derivatives not present in (6). But (6), considered as a set of simple forms, will be the basic set of a prime system, Δ, even after the adjunction of the new parametric derivatives to the unknowns in the simple forms. Following § 65, and using the passivity established above, we see that every solution of Δ which annuls no separant or initial in (6), leads, when considered at a quite arbitrary point of $\mathfrak{A}$, to a regular solution of (6). Thus $G_1 H_1$, considered as a simple form, is in Δ. Then one of G_1, H_1 is in Δ. As in § 65, it follows that one of G_1, H_1 vanishes identically. Then one of G, H is in Σ. Thus Σ is irreducible. What precedes shows that if G is in Σ, the remainder of G with respect to (6) is zero. Then (6) is a basic set of Σ.

127. Given a set (6) which satisfies the first condition of § 126, we can determine with a finite number of differentiations, rational operations and factorizations, whether or not (7) is

passive. This follows from the fact that the expressions of the μ in terms of w can be formed by a finite number of such operations.

If (7) is not passive, the form WDB_0K, (as in § 125), which involves only parametric derivatives, vanishes for any regular solutions which (6) may have.*

ALGORITHM FOR DECOMPOSITION

128. Let Σ be any *finite* system of forms, not all zero. As in § 67, we can get, by a finite number of differentiations, rational operations and factorizations, a set, equivalent to Σ, of finite systems, $\Sigma_1, \cdots, \Sigma_s$, which have the following properties:

(a) The basic sets of each Σ_i are not higher than those of Σ;

(b) if the basic sets of Σ_i involve unknowns, the remainder of any form of Σ_i with respect to a basic set is zero;

(c) a basic set of Σ_i, considered as a set of simple forms, is a basic set of a prime system.

Suppose that Σ_1 has a basic set (6), with A_1 not free of unknowns. If (7) is not passive, Σ_1 is equivalent to

$$\Sigma_1 + WDB_0K, \ \Sigma_1 + S_1, \ \cdots, \ \Sigma_1 + I_r,$$

where S_i and I_i are the separant and initial of A_i. Now all of the latter systems have basic sets lower than (6). If (7) proves passive, Σ_1 is equivalent to

$$\Omega, \quad \Sigma_1 + S_1, \quad \cdots, \quad \Sigma_1 + I_r$$

where Ω is the closed irreducible system of which (6) is a basic set.

It is clear that by this process, we arrive, in a finite number of steps, at a finite number of ascending sets, which are basic sets of a set of irreducible systems equivalent to Σ.

* It will be seen in § 129 that WDB_0K vanishes for all solutions which annul no initial.

The above constitutes a complete elimination theory for systems of algebraic partial differential equations.

The test for a form to hold a system is as in § 68.

One will notice that every system of linear partial differential equations is irreducible.

ANALOGUE OF THE HILBERT-NETTO THEOREM

129. We shall extend the theorem of § 77 to the case of several independent variables. As in the case of one variable, it suffices to show that if the system

$$(15) \qquad F_1, F_2, \cdots, F_t$$

has no solutions, then unity is a linear combination of the F_i, and of a certain number of their partial derivatives.

We suppose that unity has no such expression. One proves, as in the case of one variable, that there is a point $a_1, \cdots, a_m$, at which the coefficients in (15) are analytic, for which certain n power series

$$(16) \qquad c_{0i} + c_{1i}(x_1 - a_1) + \cdots + c_{mi}(x_m - a_m) + \cdots$$

render each F_j zero when substituted formally for $y_1, \cdots, y_n$. We shall use this fact to prove that (15) has analytic solutions.

Let (15) be resolved into irreducible systems, by the method of § 128. Here, we are dealing with analytic solutions, and not with formal ones. If we can show that one of the irreducible systems has a basic set in which the first form involves unknowns, we shall know that (15) has analytic solutions.

Let us examine the process of decomposing (15) into irreducible systems, following § 128. First, it is apparent that (16) is a formal solution of one of the systems Σ_i.* Let (16) be a solution of Σ_1. Let (6) be a basic set of Σ_1.

* The coefficients in the Σ_i may not be analytic at $a_1, \cdots, a_m$. In that case, the coefficients are to be expressed as quotients of power series for $a_1, \cdots, a_m$. This will be possible, since the coefficients are meromorphic.

Then A_1, in (6), involves unknowns. If the system (7) is passive, then (15) has analytic solutions. Suppose that (7) is not passive. We shall prove that

$$I_1 \cdots I_r WDB_0 K \tag{17}$$

vanishes for (16). Let us suppose that

$$I_1 \cdots I_r DB_0 K$$

does not vanish for (16).

The system of simple forms Ω, obtained by adjoining the simple form (see (8))

$$w - b_1 p_1 - \cdots - b_r p_r$$

to Λ of § 126, is indecomposable. We are dealing here with analytic solutions of Ω. The forms

$$\begin{gathered} B_0 w^s + \cdots + B_s, \\ Dp_i - E_{i0} - \cdots - E_{i,s-1} w^{s-1}, \quad i = 1, \cdots, r, \end{gathered} \tag{18}$$

all hold Ω. Thus, if L is any one of the $r+1$ forms (18), L vanishes for every analytic solution of the system of simple forms

$$A_1, \cdots, A_r, w - b_1 p_1 - \cdots - b_r p_r \tag{19}$$

for which $I_1 \cdots I_r$ does not vanish. By the Hilbert-Netto theorem for simple forms, some power of

$$I_1 \cdots I_r L$$

is a linear combination of the forms in (19). This means that L vanishes for any formal power series solution of (19) for which $I_1 \cdots I_r$ does not vanish.

Now, let the p_i be series obtained by differentiating (16) formally and let w be the series given by (8). We see that w satisfies (9) and that the p_i are given by (10).

If we go formally through the process of obtaining the μ of § 126, we find that the expression for every μ in terms

of w vanishes for (16). Then, if μ_1, for instance, is not identically zero, W must vanish for (16).

Thus, if Σ_1 does not have a passive system (7), (16) is a solution of one of the systems

$$\Sigma_1 + WDB_0K, \ \Sigma_1 + I_1, \ \cdots, \ \Sigma_r + I_r.$$

Continuing, we find that (16) is a solution of a basic set of some irreducible system Σ' held by (15). Then the first form of this basic set must involve unknowns, so that Σ' has analytic solutions.

This completes the proof of the analogue, for partial differential forms, of the Hilbert-Netto theorem. It follows, as in the case of one independent variable, that any finite system of forms can be decomposed into finite irreducible systems by differentiating the forms of the system a sufficient number of times and resolving the extended system, considered as a system of simple forms, into indecomposable systems.

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